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A
'EXT BOOK OF ALGEBRA

WITH

NUMEROUS MODEL SOLUTIONS AND GRADUATED EXERCISES

FOR THE USE OF STUDENTS

Preparing for the Lower and Upper Secondary, and Matriculation Examinations

BY

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PREFACE TO THE THIRD EDITION.

THE book has again been carefully revised and some slight alterations have been made. It is hoped that this edition will meet with as much encouragement from both the teacher and the taught as the two previous editions

MADRAS, January, 1898. V. R. V

PREFACE TO THE SECOND EDITION.

THE book has been carefully revised. At the suggestion of many teachers who have been using the book, answers to the University papers are also given. It is hoped that this edition will meet with as much encouragement as the first.

CALICUT, January, 1895. V. R. V.

PREFACE TO THE FIRST EDITION.

THIS work is intended as a text-book on Algebra for all classes in our High Schools which prepare their pupils either for the Government Lower and Upper Secondary Examinations or for the University Matriculation Examination. Great pains have been taken to render the work intelligible to young students by copious explanations and illustrations.

The plan adopted in this work is one which I have, for many years, followed in teaching large classes in several Schools and Colleges and which I have found to succeed. A proposition is first clearly demonstrated, then copiously illustrated by select examples, and then a large number of other examples arranged progressively is given for exercise

In addition, at the end of every eight chapters, a series of examination papers, each paper containing ten questions, is inserted.

The subject of Factors is fully treated in Chapter VIII. The University having prescribed "Quadratic Equations and Problems leading to them" for the Matriculation Examination from this year, I have devoted four Chapters (XXVI—XXIX to an exhaustive treatment of this subject. The last Chapter in the book will, it is hoped, be interesting and instructive to the more advanced student.

The book contains more than 4,000 examples, of which over 400 are fully worked out. They have been, for the most part, selected from the Examination Papers of English and Indian Colleges and Universities. At the suggestion of several friends, I have printed *all* the Entrance papers of the Universities of Calcutta, Bombay and Madras separately at the end of the book.

In the preparation of this work, most of the recent treatises on Algebra have been consulted.

Any suggestions for the improvement of the work will be thankfully received.

SAIDAPET,
January, 1892.

V. R. V.

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A
TEXT BOOK OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND EXPLANATIONS OF SIGNS

1. **Algebra**, like Arithmetic, is a science of *numbers;* but the numbers in Algebra are *generally* denoted by *letters* instead of by *figures.*

2. It is usual to represent *known* numbers by the first letters of the alphabet, as a, b, c and *unknown* numbers by the last letters, as x, y, z; but this need not be strictly followed.

The word *quantity* is often used in the sense of *number,* integral or fractional. Any *whole* number is called an *integer* or an *integral* number. Integers divisible by 2 are *even;* those that are not divisible by 2 are *odd.*

3. The sign $+$ (read *plus*) signifies that the quantity to which it is prefixed is to be *added.* Thus $a+b$ (read a plus b) means that the number denoted by b is to be added to that denoted by a. If a represents 4 and b represents 5, $a+b$ represents 9. Similarly $a+b+c$ means that b is to be added to a and that to the result c is to be added; if a, b, c, denote 4, 5, 6 respectively, $a+b+c$ denotes 15.

When no sign is prefixed to the letter at the beginning of an expression, $+$ is understood. Thus $a+b$ means $+a+b$.

4. The sign $-$ (read *minus*) signifies that the quantity to which it is prefixed is to be subtracted. Thus $a-b$ (read *a minus b*) means that the number denoted by b is to be subtracted from that denoted by a; if a denotes 7 and b denotes 4, $a-b$ denotes 3. Similarly $a-b-c$ means that b is to be subtracted from a and that from the result c is to be subtracted; if a, b, c represent 7, 4, 2 respectively, $a-b-c$ represents 1.

Quantities to which the sign + is prefixed, or before which it is understood, are called *positive* quantities.

Quantities having the sign — prefixed to them are called *negative* quantities.

5. The *difference* of two numbers is denoted by the sign $\backsim$; thus $a \backsim b$ denotes the difference of the numbers denoted by a and b, and is equal to $a-b$ or to $b-a$, according as a is greater or less than b.

Note.—The sign $\pm$ denotes *plus* or *minus*.

6. The sign $\times$ (read *into*) signifies that the numbers between which it stands are to be *multiplied* together.

Thus $a \times b$ (*a into b*) denotes that the number represented by a is to be multiplied by the number represented by b. If a represents 4 and b represents 3, then $a \times b$ represents 12.

The sign of multiplication is often omitted for the sake of brevity; thus ab is used instead of $a \times b$ and has the same meaning; so abc is used for $a \times b \times c$. Sometimes a point is used instead of the sign $\times$; thus $a.b$ is used for $a \times b$ or ab.

7. The sign $\div$ (read *by*) signifies that the number which precedes it is to be *divided* by the number which follows it. Thus $a \div b$ (*a by b*) signifies that the number denoted by a is to be divided by the number denoted by b. If a represents 10, and b represents 2, then $a \div b$ represents 5.

$\frac{a}{b}$ is used instead of $a \div b$ and has the same meaning.

N.B.—The word *sign* is often used to denote exclusively the signs + and —. Two signs are called *like* when they are both + or both — ; otherwise, they are *unlike*.

8. The sign $=$ (read *equal to*) signifies that the numbers between which it is placed are *equal*, and is called the *sign of equality*. $a=b$ is read thus, "*a equals b*," or "*a is equal to b*."

9. The sign $>$ denotes *greater than*, and the sign $<$ denotes *less than*; thus $a > b$ denotes *a is greater than b*; and $a < b$ denotes *a is less than b*. The sign $\ngtr$ denotes *not greater than* and the sign $\nless$ denotes *not less than*.

10. The sign $\therefore$ denotes *then* or *therefore*; the sign $\because$ denotes *since* or *because*.

11. A *vinculum* $\overline{}$, or a pair of *brackets* ( ) or { }, or [ ], signifies that all quantities under or within it are equally

affected by any quantity without. Thus $a(b+c)$ or $a.\overline{b+c}$ denotes that a is to be multiplied by the sum of $b+c$; whereas $a.b+c$ or $ab+c$ would signify that a was to be multiplied by b, and their product increased by c. If $a=2$; $b=3$; $c=4$; the first would $=14$; the second would $=10$.

12. The *reciprocal* of any quantity is that quantity inverted, or unity divided by it. The reciprocal of a is $\frac{1}{a}$; that of $a+b$ is $\frac{1}{a+b}$.

13. Any collection of algebraical symbols is called an *algebraical expression.*

Those parts of an expression which are connected by the signs + or — are called its *terms.*

When an expression consists of *one* term, as a, bx, $3abx$, it is called a *monomial* expression; when it consists of *two* terms, as $a \pm b$, $ax \pm by$, it is called a *binomial* expression; when it consists of *three* terms, it is called a *trinomial* expression; any expression consisting of several terms is called a *multinomial* or *polynomial* expression.

14. When an expression consists of *one* term, it is called a *simple* expression; when it consists of *more than one term*, it is called a *compound* expression.

15. When a quantity consists of the product of two or more quantities, each of the latter is called a *factor.* Thus a, b and c are factors of the product abc.

16. When a number consists of the product of two factors, each factor is called the *co-efficient* of the other factor. In the product $3a$, 3 is the co-efficient of a; where there is no arithmetical factor, we may supply unity; in the product ab, the co-efficient of ab is unity.

In the product abc, we may call a the co-efficient of bc, or b the co-efficient of ac, or c the co-efficient of ab. These co-efficients are called *literal* co-efficients as distinguished from the former which are called *numerical* co-efficients.

17. If a number be multiplied by itself any number of times, the product is called a *power* of that number. Thus $a \times a$ is called the *second power* or *square* of a; $a \times a \times a$ is called the *third power* or *cube* of a; $a \times a \times a \times a$ is called the *fourth power* of a, and so on. The number a itself is called the *first power* of a.

Any power of a quantity is usually expressed by placing above the quantity the number which represents how often it is repeated in the product. Thus a^2 (*a squared*) is used to express $a \times a$; a^3 (*a cubed*) is used to express $a \times a \times a$; a^4 is used to express $a \times a \times a \times a$, and so on. a^1 has the same meaning as a.

The numbers placed above a quantity to express the powers of that quantity are called *indices* or *exponents* of the powers.

a^n is the product of n factors each equal to a, and n is called the *index* or *exponent* of a^n (where n is any whole number).

The symbol a^4 is read thus, "a to the fourth power," or "a to the fourth" and a^n is read thus, "a to the n^{th}."

18. The *square root* of any proposed quantity is that quantity whose second power or square gives the proposed quantity.

The *cube root* of any proposed quantity is that quantity whose third power or cube gives the proposed quantity.

The m^{th} root of any quantity is that quantity whose m^{th} power gives the proposed quantity.

The square root of a is denoted by $\sqrt[2]{a}$ or simply by $\sqrt{a}$ usually.

The cube root of a is denoted by $\sqrt[3]{a}$ and the m^{th} root of a by $\sqrt[m]{a}$.

The sign $\sqrt{\ }$ is said to be a corruption of the letter r, the first letter of the word *radix*. The sign is called the *radical sign*.

$\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, . . . $\sqrt[m]{a}$ can also be represented by $a^{\frac{1}{2}}, a^{\frac{1}{3}}, a^{\frac{1}{4}}, \ . \ . \ . \ a^{\frac{1}{m}}$ respectively.

19. Algebraical quantities are said to be *like* or *unlike*, according as they contain the *same* or *different* combinations of letters.

Thus a and $5a$, $-5a^2b$ and $7a^2b$, $3a^3bc$ and $-a^3bc$ are pairs of *like* quantities; a^3 and a^2, $3ab$ and $-7a$, $3a^2b$ and $4a^3b$, of *unlike* quantities.

20. Each of the letters which occur in an algebraical product is called a *dimension* of the product, and the number of the separate letters contained in the product (as if they were all written out in full) is the *degree* of the product. Thus a^2b^2c or $a \times a \times b \times b \times c$ is of five dimensions or of the fifth degree. Numerical co-efficients should not be counted ; thus $9\,a^2b$ and ab^2 are of the same dimensions, namely, of 3 dimensions.

The *number of dimensions* of a term is the *sum* of the exponents or indices of the several factors of the term.

When all the terms of an algebraical expression are of the same dimensions, the expression is said to be *homogeneous.* Thus $4a^3b^2+3ab^4+2a^2b^3+a^2bcd$ is a homogeneous expression, for each term is of five dimensions.

21. The square root or any even root of a negative quantity is called an *impossible* or *imaginary* quantity. Thus $\sqrt{-3}$ and $\sqrt{-a^2}$ are impossible or imaginary quantities.

22. When any root of a quantity cannot be exactly extracted, the quantity which represents the root is called a *surd* or an *irrational* quantity. Thus $\sqrt{3}$ and $\sqrt{ab}$ are surds.

NUMERICAL VALUES.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=0$, find the values of :—

Ex. 1. $a-2b+3c+4d-6e$
$=1-2\times2+3\times3+4\times4-6\times0=1-4+9+16-0$
$=1+9+16-4=26-4=22.$

Ex. 2. $7abc-5bcd+4cde+3acd-10ace$
$=7\times1\times2\times3-5\times2\times3\times4+4\times3\times4\times0+3\times1\times3\times4$
$-10\times1\times3\times0=42-120+48\times0+36-30\times0=42-120$
$+0+36-0=42+36-120=70-120=-42.$

EXERCISE 1 (A).

If $a=4$, $b=3$, $c=5$, $d=6$, $e=1$, $f=0$, find the numerical values of :—

1. $12c+3a-5b-2f+4e.$
2. $-6af+abc-2cd+e.$
3. $4ab+3cf-6d+7be.$
4. $abcd-bce+fac.$
5. $\frac{1}{2}ab-3bcd+5ef-6cde.$
6. $4abcd-7ade-2afe+2bcde.$
7. $4ac-9ae+11bcd-10afc.$
8. $42a-21bc-14ae+36ace.$
9. $20c-\frac{1}{2}cde+\frac{1}{4}abd-2cd.$
10. $12b-6cf+7cde-9abde.$

Ex. 3. Find the value of :—

$\frac{2d}{a}-\frac{6e}{b}-\frac{15a}{cd}+\frac{4b}{d}$ (for values of a, b, &c., as in Example 1)

$$=\frac{2\times 4}{1}-\frac{6\times 0}{2}-\frac{15\times 1}{3\times 4}+\frac{4\times 2}{4}=8-0-\frac{5}{4}+2=10-\frac{5}{4}=8\frac{3}{4}.$$

Ex. 4. If $a=4$, $b=3$, $c=2$, find the value of :—

$ab(a-b)-ac(a-c)+bc(b-c)=4\times 3(4-3)-4\times 2(4-2)$
$+3\times 2(3-2)=12\times 1-8\times 2+6\times 1=12-16+6=18$
$-16=2.$

EXERCISE 1 (B).

If $a=3$, $b=4$, $c=6$, $d=5$, $e=1$, $f=0$, find the values of :—

1. $\dfrac{a+b}{c}+\dfrac{b+c}{a}+\dfrac{c+a}{b}$.
2. $\dfrac{a}{b-e}+\dfrac{b}{a-b}+\dfrac{c}{d-c}$.
3. $ab\div cd-bd\div ac$.
4. $a-b(d-e)+c(a+b)$.
5. $\{a+(b+c)-d\}\ \{(a-b)-(c-d)\}\ (a-\overline{b+c+d})$.
6. $\{12a-(3b-3e)\}-\{4f-\overline{4b-3d+5c}\}$.
7. $\dfrac{ab}{cd}-\dfrac{bc}{a}+\dfrac{ac}{e}+\dfrac{abc}{d}-\dfrac{cde}{b}$.
8. $a(b-c)+b(c-a)+c(a-b)+d(a-e)$.
9. $4a-e(2b-3c)+d(b-2c)+a(2d-3e)$.
10. $\dfrac{a+b}{b+c}+\dfrac{b+c}{c+a}+\dfrac{c+a}{a+b}$.

Ex. 5. If $a=1$, $b=2$, $c=3$, find the value of :—

$(a+b)^2(a^2-ab+b^2)+(a+c)^2(a^2-ac+c^2)=(1+2)^2$
$(1^2-1\times 2+2^2)+(1+3)^2(1^2-1\times 3+3^2)=3^2(1-2+4)$
$+4^2(1-3+9)=9\times 3+16\times 7=27+112=139.$

Ex. 6. If $a=2$, $b=3$, $x=6$, $y=5$, find the value of :—

$\sqrt[3]{\{(a+b)^2y\}}+\sqrt[3]{\{(a+x)(y-2a)\}}+\sqrt[3]{\{(y-b)^2a\}}$
$=\sqrt[3]{\{(2+3)^2 5\}}+\sqrt[3]{\{(2+6)(5-2\times 2)\}}+\sqrt[3]{\{(5-3)^2 2\}}$
$=\sqrt[3]{\{5^2\times 5\}}+\sqrt[3]{\{8\times 1\}}+\sqrt[3]{\{2^2\times 2\}}$
$=\sqrt[3]{5^3}\times\sqrt[3]{8}+\sqrt[3]{2^3}=5+2+2=9.$

EXERCISE 1 (C).

If $a=5$, $b=2$, $c=1$, $d=4$, $e=8$, $f=0$, $m=6$, $n=4$, $p=9$, find the values of :—

1. $\sqrt{2a-1}+\sqrt[3]{an+p-b}-\sqrt{a^2-d^2}$.

2. $(a+b)^2-(d-c)^2+(f+e)^2-(p-m+n)^2$.

3. $\sqrt{p}-\sqrt{n}+\sqrt[3]{e}-\sqrt{c}+\sqrt{cd}-\sqrt[3]{ef}$.

4. $\dfrac{a+b+c}{m+n}-\dfrac{b+c+d}{n+p}+\dfrac{c+d+e}{m+p}-\dfrac{d+e+f}{m+n+p}$.

5. $a^c-b^d+d^b-m^b+n^c-p^b$.

6. $\sqrt{pn-df-p-b}+\sqrt[n]{m^2-pd}$.

7. Find the value of $(9-a)(b+1)+(b+5)(a+7)-112$ when $a=5$ and $b=3$.

8. If $a=16$, $b=10$, $x=5$ and $y=1$, find the value of $(b-x)\times\sqrt{(a+b)}+\sqrt{\{(a-b)(x+y)\}}$ and of $(a-y)\{\sqrt{(2bx)}+x^2\}+\sqrt{\{(a-x)(b+y)\}}$.

9. Find the value of $a+b\sqrt{(x+y)}-(a-b)\sqrt[3]{(x-y)}$, when $a=10$, $b=8$, $x=12$ and $y=4$.

10. If $x=5$ and $a=8$, find the value of $a\sqrt{(x^2-3a)}+x\sqrt{(x^2+3a)}+\dfrac{x+2a}{a-x}$.

CHAPTER II.

ADDITION.

23. When two or more algebraical quantities are united together, the result is called their *Sum* and the process of finding the result is called *Addition.* An *algebraic sum* is not necessarily the sum of positive quantities.

24. When the terms of an expression are connected by the sign +, we may write the terms *in any order.* Thus $a+b$ and $b+a$ will give the same result.

When an expression consists of some terms preceded by the sign + and some terms preceded by the sign —, we may write the former terms first in any order we please, and the latter terms after them in any order we please. Thus $a+d-b-c=a+d-c-b=d+a-c-b=d+a-b-c$.

25. There are three cases of Addition :—

First Case. Where the quantities are *like,* and have *like* signs.

Rule.—" Add the co-efficients together, and to their sum adjoin the letters common to each term, prefixing the common sign."

Ex. 1. Add ogether $3a$, $5a$, $9a$ and $7a$.
$3+5+9+7=24$ ∴ required sum $=24a$.

Ex. 2. Add together $-2b$, $-4b$, $-5b$ and $-6b$.
$2+4+5+6=17$ ∴ required sum $=-17b$.

Ex. 3. Add together $4a-3b$, $5a-6b$ and $2a-7b$.

$$\begin{array}{lcl} 4a-3b & \qquad & 4+5+2=11 \\ 5a-6b & & -3-6-7=-16 \\ \underline{2a-7b} & & \\ 11a-16b & & \end{array}$$

Ex. 4. Add together $\frac{3}{2}ax-b$, $\frac{1}{2}ax-\frac{3}{4}b$ and $\frac{1}{4}ax-\frac{1}{2}b$.

$$\begin{array}{lcl} \frac{3}{2}ax-b & \qquad & \frac{3}{2}+\frac{1}{2}+\frac{1}{4}=\frac{9}{4} \\ \frac{1}{2}ax-\frac{3}{4}b & & -1-\frac{3}{4}-\frac{1}{2}=-\frac{9}{4} \\ \underline{\frac{1}{4}ax-\frac{1}{2}b} & & \\ \frac{9}{4}ax-\frac{9}{4}b & & \end{array}$$

EXERCISE 2.

Add together :—

1. $2a, 5a, a$; $-2b, -3b, -b$; $\frac{1}{2}a, \frac{1}{4}a, \frac{5}{4}a$.
2. $10ab, 2ab, 9ab, 16ab$; $6a^2b, 4a^2b, 3a^2b$.

3. $4\sqrt{xy}$, $3\sqrt{xy}$, $9\sqrt{xy}$, $\sqrt{xy}$; $-\sqrt{ab}$, $-2\sqrt{ab}$, $-\frac{5}{2}\sqrt{ab}$.
4. $2a-3b$, $4a-10b$, $a-2b$, $2a-b$, $14a-13b$.
5. a^2-ab+b^2, $2a^2-3ab+3b^2$, $3a^2-4ab+7b^2$.
6. $x^2-2xy+y^2$, $\frac{1}{2}x^2-\frac{1}{4}xy+2y^2$, $\frac{3}{4}x^2-xy+\frac{5}{2}y^2$.
7. $\frac{1}{2}-ax+by-ab$; $2\frac{1}{2}-3ax+\frac{1}{2}by-3ab$, $4\frac{1}{2}-\frac{4}{3}ax+3by-\frac{3}{2}ab$, $4-10ax+by-\frac{3}{2}ab$.

26. *Second Case.* Where the quantities are all *like*, but the signs all *unlike*.

Rule.—"Add all the positive co-efficients into one sum, and all the negative co-efficients into another; subtract the less of these sums from the greater, and to the difference prefix the sign of the greater sum, and then adjoin the common letters. If the aggregate of the positive terms be equal to that of the negative ones, then the difference $=0$."

Ex. 5. Add together :—

$4x^2+3xy+y^2$, $-3x^2+xy-2y^2$ and $2x^2-5xy+3y^2$.

Proceeding by the rule,

$$\begin{array}{rrr} 4x^2 & +3xy & +y^2 \\ -3x^2 & +xy & -2y^2 \\ 2x^2 & -5xy & +3y^2 \\ \hline 3x^2 & -xy & +2y^2 \end{array} \qquad \begin{array}{l} 4+2-3=3 \\ 3+1-5=-1 \\ 1+3-2=2 \end{array}$$

Ex. 6. Add together :—

$\frac{1}{2}a+\frac{1}{3}b-\frac{1}{6}c$, $\frac{1}{3}a-\frac{1}{6}b+\frac{1}{3}c$,
$\frac{1}{6}a-\frac{1}{2}b-\frac{2}{3}c$ and $a-b+c$.

Proceeding by the rule,

$$\begin{array}{rrr} \frac{1}{2}a & +\frac{1}{3}b & -\frac{1}{6}c \\ \frac{1}{3}a & -\frac{1}{6}b & +\frac{1}{3}c \\ \frac{1}{6}a & -\frac{1}{2}b & -\frac{2}{3}c \\ a & -b & +c \\ \hline 2a & -\frac{4}{3}b & +\frac{1}{2}c \end{array} \qquad \begin{array}{l} \frac{1}{2}+\frac{1}{3}+\frac{1}{6}+1=2 \\ \frac{1}{3}-\frac{1}{6}-\frac{1}{2}-1=-\frac{4}{3}. \\ -\frac{1}{6}+\frac{1}{3}-\frac{2}{3}+1=\frac{1}{2} \end{array}$$

EXERCISE 3.

Add together :—

1. $a+b+c+d$, $-a+b+c+d$, $a-b+c+d$, $a+b-c+d$ and $a+b+c-d$.
2. $a^4+a^3-a^2+1$, $a^3-2a^4+3a^2-3$ and $4a^2-3a^3-4a^4+4$.

3. $ax+by+cz$, $-3ax-2by-3cz$ and $4ax-3by+4cz$.

4. $a^4+4a^3+6a^2+4a+1$ and $a^4-4a^3+6a^2-4a+1$.

5. $a^3-\frac{3}{4}a^2x+\frac{1}{3}ax^2-\frac{5}{8}x^3$, $a^3+\frac{1}{2}a^2x+\frac{1}{4}ax^2+\frac{2}{3}x^3$ and $2a^3-\frac{1}{4}a^2x+\frac{7}{12}ax^2+\frac{1}{24}x^3$.

6. $a^m-2b^n+3c^p-4d$, $23a^m+16b^n-13c^p+12d$, $-14a^m+15b^n-17c^p+19d$, $18a^m-b^n+c^p-d$ and $-13a^m+5b^n-3c^p+11d$.

27. *Third Case.* Where the quantities are *unlike*.

Rule.—"Set all the quantities down one after another with their co-efficients and proper signs prefixed, and collect all the like quantities together (if there be any) by the foregoing rules."

Ex. 7. Add together:—

$ab+ac+bc$, $ax-2ab+by$, $bz-2ax+cy$ and $bc-cz+3ac$.

Proceeding by the rule,

$$
\begin{array}{llllllll}
ab & +ac & +bc & & & & & \\
-2ab & & & +ax & +by & & & \\
 & & & -2ax & & +bz & +cy & \\
 & +3ac & +bc & & & & & -cz \\
\hline
-ab & +4ac & +2bc & -ax & +by & +bz & +cy & -cz
\end{array}
$$

EXERCISE 4.

Add together:—

1. $a-x-y$, $b-c-d$, $x+y+c$ and $d+c-x-y$.

2. ay^2+by+c, by^2+cy+a and cy^2-by-a.

3. $x^2+xy+xz$, $y^2+xy+yz$, $z^2+zx+zy$ and $x^2+y^2+z^2-2(xy+yz+zx)$.

4. $3ab^2-4a^2b+a^3$, $-4ac^2+5ab^2-c^3$, $-7b^3+2a^2b-6ac^2$ and $5a^3-11ab^2-12ac^2$.

5. $a-\frac{1}{3}b+\frac{1}{4}c-\frac{1}{5}d$, $-\frac{1}{2}c+\frac{1}{3}a-\frac{1}{4}b+d$, $\frac{1}{4}d-\frac{1}{5}b+c-a$, $\frac{1}{6}a-\frac{3}{5}d+b-\frac{5}{8}c$ and $8a-6b+3c-4d$.

6. $a-\frac{b}{2}+\frac{c}{3}-\frac{d}{4}$, $-\frac{b}{3}-\frac{a}{4}+d-\frac{c}{4}$ and $\frac{a}{5}-\frac{b}{4}+\frac{c}{3}-\frac{d}{2}$.

7. $1+x+x^2+x^3+x^4$, $1-2x+3x^2-4x^4$, $x+3x^3+5x^2-7x^4$ and $1-x^4$.

8. $2x^2-6xy-\frac{28}{11}y^2$, $4y^2-\frac{4}{3}y^3+2z^2$, $2xy-\frac{2}{3}y^2+2y^3$ and $4xy-\frac{2}{3}y^2-3x^2-3y^2-3z^2$.

CHAPTER III.

SUBTRACTION AND REMOVAL OF BRACKETS.

28. Any quantity B is said to be subtracted from any other quantity A when a third quantity C is found such that the *sum* of B and C is equal to A.

i.e., $C = A - B$ when C is such that $B + C = A$.

The quantity *from* which another quantity is subtracted is called the *minuend.*

The *quantity subtracted* is called the *subtrahend.*

The result is called the *difference* or the *remainder.*

A is the minuend, B the subtrahend and C the remainder.

29. *Rule.*—"Change the signs of all the quantities to be subtracted or conceive them to be changed, and then collect the different terms together as in addition."

Ex. 1.* Subtract $3x-5b$ from $4x-8b$. Changing the sign of $3x-5b$, we have, $-3x+5b$. Add $-3x+5b$ to $4x-8b$ thus

$$\begin{array}{r} 4x-8b \\ -3x+5b \\ \hline x-3b \end{array}$$

Ex 2. Subtract $x^2-2xy+y^2$ from $x^2+2xy+y^2$.

$$\begin{array}{rrr} x^2 & +2xy & +y^2 \\ x^2 & -2xy & +y^2 \\ - & + & - \\ \hline & 4xy & \end{array}$$

Ex. 3. Subtract $2a^2+3ab-5b^2$ from $-3a^2+2ab-4b^2$.

$$\begin{array}{rrr} -3a^2 & +2ab & -4b^2 \\ 2a^2 & +3ab & -5b^2 \\ - & - & + \\ \hline -5a^2 & -ab & +b^2 \end{array}$$

---

* From $2a$ take b, and the difference is denoted by $2a-b$; because the sign $-$ prefixed to b shows that it is to be subtracted from the other and $2a-b$ is the sum of $2a$ and $-b$.

Again, from $2a$ take $-b$ and the difference is $2a+b$; because $2a = 2a+b-b$.

Take away $-b$ from these equal quantities.

$\therefore$ the difference between $2a$ and $-b$ is $2a+b$, *i.e.*, the sum of $2a$ and $+b$.

$\therefore$ to take $-b$ is the same as to add $+b$.

Ex 4. Subtract $a^2-4ab+5x^2-y^2$ from $6x^2+3y^2-6a^2$.

$$\begin{array}{l} 6x^2+3y^2-6a^2 \\ 5x^2-\ y^2+\ a^2-4ab \\ -\quad\ +\quad\ -\quad\ + \\ \hline x^2+4y^2-7a^2+4ab \end{array}$$

EXERCISE 5.

Subtract :—

1. $a-b+c$ from $3a-3b+c$.
2. $-x-y+z$ from $3x+4y-5z$.
3. $-4a^2-2ab+6$ from $-5a^2-ab+8$.
4. $2x^3+3x^2+4x+5$ from x^3-4x^2+6x+7.
5. $a^3+3a^2b+3ab^2+b^3$ from $a^3-3a^2b+3ab^2-b^3$.
6. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ from $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ and find the value of the difference, when $a=2$, and $b=3$.
7. From $\frac{1}{2}x-\frac{13}{6}a+\frac{1}{3}b-\frac{1}{4}y$ take $\frac{1}{6}x+\frac{11}{4}a-\frac{3}{8}b-\frac{1}{2}y$.
8. From $\frac{3}{2}a+3b+\frac{1}{4}c+\frac{1}{2}bc$ take $\frac{5}{2}a-7b-3bc+\frac{1}{2}c$.
9. From $\frac{1}{2}y-\frac{5}{2}a-\frac{3}{4}x+\frac{1}{3}a$ take $3y+\frac{11}{4}a-\frac{2}{3}x$.
10. From $10a^m-14b^n-c^p-5d$ take $-9a^m+2b^n+c^p-5d$.

11. From the sum of the first four of the following expressions $x^2+y^2+z^2+a^2$, $a^2+y^2+z^2$, $x^2-z^2+y^2-a^2$, $x^2-y^2+z^2+a^2$, $y^2+z^2+a^2-x^2$, subtract the sum of the last four.

12. What must be added to $3a^2-5ab+6b^2+7bc$ in order that the sum may be $-a^2-b^2-bc$?

13. What must be added to $-5a^3+13a^2b^2-x^2ya+5ab^2+7abxy$ in order that the sum may be $a^3+a^2b^2+x^2ya-2yab^2-2xyab$?

14. What must be subtracted from $4a^2-3ab-6b^2$ in order that the remainder may be a^2+3b^2 ?

15. What must be subtracted from $10x^2-7xy+5y^2-9x-11y+14$ in order that the remainder may be $x^2+xy+y^2+x+y+1$?

16. From what expression must $3x^2-7xy-8yz+9y^2$ be subtracted in order that the remainder may be $2x^2+3xy+3yz+2y^2$?

30. To prove that (1) $a-(b+c)=a-b-c$.
(2) $a-(b-c)=a-b+c$.
(3) $a-(b-c-d+e)=a-b+c+d-e$.

(1) As each of the numbers b and c is to be taken from a, the result is denoted by $a-b-c$. We enclose the term $b+c$ in brackets, because *both* the numbers b and c are to be taken from a. Therefore $a-(b+c)=a-b-c$.

(2) If we take b from a, we obtain $a-b$; but we have thus taken too much from a, for we are required to take, not b but, b diminished by c. Hence we must increase the result by c; thus $a-(b-c)=a-b+c$.

(3) We have to take $b-c-d+e$ from a. This is the same thing as taking $b+e-c-d$ from a. Take away $b+e$ from a, and the result is $a-b-e$; then *add* $c+d$, because we were to take away not $b+e$ but, $b+e$ diminished by $c+d$; thus $a-(b-c-d+e)=a-b-e+c+d=a-b+c+d-e$.

31. The following rules must be observed in the *removal of brackets* :—

(1) If any number of terms be enclosed within a pair of brackets preceded by the sign $+$, the brackets may be removed *without* altering the signs of the terms. Thus $a+(b+c-d-e)=a+b+c-d-e$.

(2) If any number of terms be enclosed within a pair of brackets preceded by the sign $-$, the brackets can be removed only if the signs of all the terms within be changed, $+$ to $-$ and $-$ to $+$. Thus $a-(b+c-d-e)=a-b-c+d+e$.

The following rules must be observed in the *insertion of brackets* :—

(1) Any number of terms may be put within brackets with the sign $+$ prefixed, without any other changes. Thus $a+b-2c=a+(b-2c)$.

(2) If the sign $-$ be prefixed, the signs of all the terms within the brackets must be reversed. Thus $a-b+2c=a-(b-2c)$.

Note.– If there be brackets within-brackets in an expression, it is the safest method to remove the innermost brackets first, then the next, and so on.

Ex. 1. Simplify $\{a-(b-c)\}-\{b-(c-a)\}$.

Given expression $=\{a-b+c\}-\{b-c+a\}$
$=a-b+c-b+c-a=2c-2b$.

Ex. 2. Simplify $a-[b-\{c-(d-e)\}-f]$.

$$\begin{aligned}\text{Given expression} &= a-[b-\{c-d+e\}-f]\\ &= a-[b-c+d-e-f]\\ &= a-b+c-d+e+f.\end{aligned}$$

Ex. 3. Simplify $7a-\{3a-[4a-(5a-\overline{2a+a})]\}$.

$$\begin{aligned}\text{Given expression} &= 7a-\{3a-[4a-(5a-2a-a)]\}\\ &= 7a-\{3a-[4a-5a+2a+a]\}\\ &= 7a-\{3a-4a+5a-2a-a\}\\ &= 7a-3a+4a-5a+2a+a\\ &= 14a-8a=6a.\end{aligned}$$

Ex. 4. Enclose $2a+b-c+2d-3e-4f$ in brackets

(1) taking the terms two together each bracket being preceded by the sign —.

(2) taking the terms three together each bracket being preceded by the sign —.

(1) $2a+b-c+2d-3e-4f=-c+2a-3e+b-4f+2d$
$= -(c-2a)-(3e-b)-(4f-2d)$.

(2) $2a+b-c+2d-3e-4f=-c+2a-3e-4f+b+2d$
$= -(c-2a+3e)-(4f-b-2d)$.

Introducing a pair of inner brackets in each, $=-\{c-\overline{2a-3e}\}$
$-\{4f-\overline{b+2d}\}$.

EXERCISE 6.

Remove the brackets from :—

1. $2a-\{2a-(b+2x)\}+\{b-(2x-2b)\}$.
2. $\{6a+2b-(3a+2b)\}-\{2a+4b-(4a-b)\}$.
3. $a-[2b-\{3c-(a-\overline{2b-3c})\}]$.
4. $a^2-(b^2-c^2)-b^2+(c^2-a^2)+c^2-(b^2-a^2)$.
5. $-a-[-b-\{-c-(-a-\overline{b-c})\}]$.
6. $11x-[7x-\{8x-(9x+\overline{2x-6x})\}]$.
7. $2-\{4-(6-\overline{7-9})\}$.
8. $-2-[-3-\{-4-(-5-\overline{6+2})\}]$.
9. $a-[5b-\{a-(3c-3b)+(2c-\overline{a-2b-c})\}]$.
10. $7a-[2a-\{b-(3a-\overline{5a-2b})-4a\}-2b]$.
11. $-[+\{+(-x)\}]-\{+[-(-x)]\}$.

12. $-3x-[-5y-\{-7z-(-9x-\overline{-11y-13z})\}]$.

13. $-2x-[-4y-\{-5z-(-2x+\overline{-5y-8z})\}]$.

14. $-x-[-2y-\{3z-(4x-\overline{-2y-9z})\}]$.

15. $-a-[-b-\{-c-(a-\overline{-b-c})\}]$.

16. $-4a-[-3b-\{-4c-(3a+\overline{-4b-5c})\}]$.

Add together :—

17. $a-(d-2c)$, $3a-(b-4c)+2d$ and $3a-b-(2c-d)$.

18. $1-\{1-(1-x)\}$, $2x-(3-5x)$ and $2-(-4+5x)$.

19. $a-(2d-3c)$, $4a-(b-4c)+3d$ and $3a-\{(2b+2c)-d\}$.

20. $(ax-bx)-(by-cy)$, $ax+bx-(by+cy)$ and $(ax-by)-(bx-cy)$.

21. In the expression $a-2b+3c-4d+5e-6f+7g-8h$.

Enclose the 1st, 3rd, 7th and 8th terms within brackets preceded by the sign $-$ and the other terms within brackets preceded by the sign $+$.

22. Include the first three in brackets preceded by the sign $-$, the next two in brackets preceded by the sign $+$, and the last three in brackets preceded by the sign $-$.

23. In the last question, introduce inner brackets in the first and the third preceded by the sign $-$.

24. Enclose in pairs the terms of $-a-b+c-d-e+f$ in brackets, so that each bracket shall be preceded by the sign $-$.

CHAPTER IV.

MULTIPLICATION.

32. The number multiplied is called the *multiplicand.* The number by which it is multiplied is called the *multiplier.* The result is called the *product.*

33. Rule of Signs.—"The sign of the product of any two quantities is positive or negative according as the multiplicand and the multiplier have like or unlike signs." More briefly—"Like signs produce + and unlike signs —."*

Note 1.—Since $(+x)\times(+x)=x^2$ and also $(-x)\times(-x)=x^2$, we have $\sqrt{x^2}=-x$. Every algebraical quantity has got two square roots equal in value but opposite in sign.

Note 2.—The product of any number of factors is *positive* or *negative* according as an *even* or *odd* number of the factors is *negative.* An *even* power of a negative quantity is *positive*, and an *odd* power is negative. Thus:—$a\times+b\times-c=+abc$; $-a\times-b\times-c=-abc$; $(-a)^4=a^4$; and $(-a)^7=-a^7$.

34. To multiply *simple* algebraical quantities:—

Rule.—"Multiply together the numerical co-efficients and write the letters after the product of the numbers, prefixing to the product the proper sign."

Ex. 1. $5a\times 2b=10ab$ ($\because$ + into + gives +).

Ex. 2. $15x^2\times-4y^2=-60x^2y^2$ (+ into − gives −).

Ex. 3. $-\frac{1}{2}pq\times-\frac{1}{4}mn=\frac{1}{8}pqmn$ (− into − gives +).

35. To multiply two or more *powers of the same quantity*:—

Rule.—"Add the indices of the powers of the proposed quantity and write the quantity with an index equal to the sum."

Ex. 4. $a^3\times a^2=a^{3+2}=a^5$; $x^4\times x^8=x^{4+8}=x^{12}$ and $a^m\times a^n=a^{m+n}$.

To prove that $a^m\times a^n=a^{m+n}$ when m and n are positive integers.

Since $a^m=a\times a\times a$ . . . . to m factors.
and $a^n=a\times a\times a\times a$ . . . . . . to n factors.
$\therefore$ $a^m\times a^n=a\times a\times a\times a$ . . . to $m+n$ factors.
$=a^{m+n}$.

---

* The fact that — into — gives + may be explained thus: the sign — is the symbol of reversal; if a thing is *twice reversed,* it is necessarily restored to the original condition.

Similarly $a^m \times a^n \times a^p = a^{m+n} \times a^p$
$= a^{m+n+p}.$

To prove that $(a^m)^n = a^{mn}$ when m and n are positive integers $(a^m)^n = a^m \times a^m \times a^m$ to n factors
$= a^{m+m+m}$ to n terms
$= a^{mn}.$

The rule holds good also when m and n are *negative, integral* or *fractional.*

Thus: $a^{-m} \times a^{-n} = a^{-m-n}$; $a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}$

$a^{-m} \times a^{\frac{p}{q}} = a^{-m+\frac{p}{q}}$; $a^{-m} \times a^n = a^{n-m}$

EXERCISE 7 (A).

Multiply :—

1. $6ab$ by $8c$; $4a$ by $-5y$; $5a^3$ by $-7a^2$; $4x^3y$ by $3x^2y^2$.
2. $-5a^3bc$ by $2ab^2c^2$; $10x^2yz$ by $-7x^3y^4z^2$; $-18xyz$ by $5x^2z^2$.
3. $4p^2q^2$ by $-7apq$; $-8a^3xy$ by $-6ab^3x^3y^2$.
4. $7x^2 - 2x + 4a$ by $3a^2$; $4x^2y - 2xy + 4x - 3y$ by $-2xy$.
5. $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{4}$ by $-\frac{2}{3}x^2$.
6. $7x^4 - 3x^3y - 4x^2y^2 + 8xy$ by $\frac{1}{2}x^2y^2$.
7. $4a^2b - 5a^2b^2 + 6ab^3 - 2b^4$ by $-\frac{1}{3}ab$.
8. $\frac{1}{2}p^2q^2 - \frac{1}{3}p^3q + \frac{1}{4}pq^2 - q^3$ by $\frac{1}{4}p^2q$.
9. $10x^3y - \frac{4}{5}x^2y^2 + \frac{3}{4}xy^3 - \frac{1}{2}y^4$ by $2xy^2$.
10. $\frac{5}{2}x - \frac{2}{3}y^2 + \frac{4}{7}x^3y - \frac{2}{3}x^1y^2 - 4x^3y^3 + \frac{2}{7}y^3$ by $-\frac{2}{3}x^2y$.

36. To multiply *compound algebraical quantities.*—

Rule.—"Multiply every term of the multiplicand by each term of the multiplier; then connect the several products together by the rules of addition, and that sum will be the product required."

Thus: $a \times (b \pm c) = ab \pm ac.$
$(a+b)(c+d) = ac + ad + bc + bd.$
$(a-b)(c-d) = ac - ad - bc + bd.$

Note.—When factors are multiplied together, the product is the same in whatever order the operation is performed.

Thus: $a \times b = b \times a$ and $a \times b \times c = b \times c \times a$.

Ex. 3. Multiply $(a+b)$ by $(a+b)$.

$$(a+b)(a+b) = a \times a + a \times b + b \times a + b \times b$$
$$= a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2.$$

Ex. 2. Multiply $a-b$ by $a-b$.

$$\begin{aligned}(a-b)(a-b)&=a\times a+(-b)\times a+a\times(-b)+(-b)(-b)\\&=a^2+(-ab)+(-ab)+b^2\\&=a^2-2ab+b^2.\end{aligned}$$

Ex. 3. Multiply $a+b$ by $a-b$.

$$\begin{aligned}(a+b)(a-b)&=a\times a+b\times a+a\times(-b)+b\times(-b)\\&=a^2+ba-ab-b^2\\&=a^2-b^2.\end{aligned}$$

Ex. 4. Multiply $2x+3y$ by $4x-5y$.

$$\begin{array}{l}2x+3y\\ \underline{4x-5y}\\ 8x^2+12xy\\ \underline{\qquad-10xy-15y^2}\\ 8x^2+\ \ 2xy-15y^2.\end{array}$$

Ex. 5. Multiply $3a^2-4a+8$ by $2a+3$.

$$\begin{array}{l}3a^2-4a+8\\ \underline{2a\ +3}\\ 6a^3-8a^2+16a\\ \underline{\qquad+9a^2-12a+24}\\ 6a^3+\ a^2+\ 4a+24.\end{array}$$

Ex. 6. Multiply a^2+ab+b^2 by $a-b$.

$$\begin{aligned}(a-b)(a^2+ab+b^2)&=a(a^2+ab+b^2)+(-b)(a^2+ab+b^2)\\&=a\times a^2+a\times ab+a\times b^2+(-b)\times a^2+(-b)\times ab\\&\quad+(-b)\times b^2\\&=a^3+a^2b+ab^2-ba^2-ab^2-b^3\\&=a^3-b^3.\end{aligned}$$

Ex. 7. Multiply a^2-ab+b^2 by $a+b$.

$$\begin{array}{l}a^2-ab+b^2\\ \underline{a\ +b}\\ a^3-a^2b+ab^2\\ \underline{\qquad+a^2b-ab^2+b^3}\\ a^3\qquad\qquad\quad+b^3.\end{array}$$

EXERCISE 7 (B).

Multiply :—

1. $x+a$ by $x-a$; $x+2a$ by $x+3a$; $x+4$ by $x+9$; a^2+ab+b^2 by $a+b$.
2. $2a+b$ by $a+b$; $2m-3n$ by $m-n$; x^2+y^2 by x^2-y^2.

3. $a+b+c$ by $a+b+c$; $a-b+c$ by $a+b-c$.
4. $xy+yz+zx$ by $xy-yz-zx$; $x^2+y^2+z^2$ by $x-y-z$.
5. $a-3b+4c$ by $a+2b-3c$; $m+n+p$ by $3m-4n-5p$.
6. $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.
7. $x^4+x^3+x^2+x+1$ by $x-1$.
8. $a^3+a^2b+ab^2+b^3$ by $a-b$.
9. $4a^2-3a+7$ by $5a^2-4a+8$.
10. $a^2+ab+ac+bc$ by $b+c$.

37. When one expression is to be multiplied by another, it is convenient to arrange both the multiplicand and the multiplier according to descending or ascending powers of some letter common to them and then proceed with the multiplication.

The expression $2a+3a^3-6a^2+a^4+a^5+2$ will be arranged as $a^5+a^4+3a^3-6a^2+2a+2$ in *descending* powers of a and as $2+2a-6a^2+3a^3+a^4+a^5$ in *ascending* powers of a.

Ex. 1. Multiply $ab+b^2+a^2$ by a^2+b^2-ab.

$$\begin{array}{rl}
\text{Multiplicand} & =a^2+ab+b^2 \\
\text{Multiplier} & =a^2-ab+b^2 \\ \hline
a^2(a^2+ab+b^2) & =a^4+a^3b+a^2b^2 \\
-ab(a^2+ab+b^2) & = \quad\;\; -a^3b-a^2b^2-ab^3 \\
b^2(a^2+ab+b^2) & = \qquad\qquad\quad +b^2a^2+ab^3+b^4 \\ \hline
\therefore \text{the product} & =a^4+a^2b^2+b^4.
\end{array}$$

Ex. 2. Multiply $2x^2+3xy+4y^2$ by $2x^2-3xy+4y^2$. Arranging the multiplicand and the multiplier according to the descending powers of y, we have,

$$\begin{array}{rrrrr}
4y^2 & +3yx & +2x^2 & & \\
4y^2 & -3yx & +2x^2 & & \\ \hline
16y^4 & +12y^3x & +8y^2x^2 & & \\
 & -12y^3x & -9y^2x^2 & -6yx^3 & \\
 & & +8y^2x^2 & +6yx^3 & +4x^4 \\ \hline
16y^4 & & +7y^2x^2 & & +4x^4
\end{array}$$

Ex. 3. Multiply ax^2+bx+c by cx^2+ax+b.

$$\begin{array}{lllll}
ax^2 & +bx & +c & & \\
cx^2 & +ax & +b & & \\ \hline
acx^4 & +bcx^3 & +c^2x^2 & & \\
 & a^2x^3 & +abx^2 & +acx & \\
 & & abx^2 & +b^2x & +bc
\end{array}$$

$$\text{Product} = acx^4+(cb+a^2)x^3+(c^2+2ab)x^2+(ac+b^2)x+bc.$$

Ex. 4. Multiply px^3+qx^2+rx+s by ax^2-bx-c

$$\begin{array}{l}
px^3+qx^2+rx+s \\
\qquad ax^2-bx-c \\
\hline
apx^5+aqx^4+arx^3+asx^2 \\
\qquad -pbx^4-bqx^3-brx^2-bsx \\
\qquad\qquad -cpx^3-cqx^2-crx-cs \\
\hline
\end{array}$$

$$\text{Product}=apx^5-(pb-aq)x^4+(ar-bq-cp)x^3-(br+cq-as)$$
$$\times x^2-(bs+cr)x-cs$$

Note—The product in Examples 3 and 4 is put within brackets

Ex. 5 Find the continued product of $x+1$, $x-1$ and $x+2$

$$\begin{aligned}
(x+1)(x-1)&=x(x-1)+1(x-1) \\
&=x^2-x+x-1=x^2-1 \\
\text{and }(x+2)(x^2-1)&=x(x^2-1)+2(x^2-1) \\
&=x^3-x+2x^2-2=x^3+2x^2-x-2
\end{aligned}$$

Ex 6 Find the continued product of $x+a$, $x+b$ and $x+c$

$$\begin{array}{l}
x+a \\
x+b \\
\hline
x^2+xa \\
\quad +bx+ab \\
\hline
x^2+(a+b)x+ab
\end{array}
\qquad\qquad
\begin{array}{l}
x^2+(a+b)x+ab \\
x+c \\
\hline
x^3+(a+b)x^2+abx \\
\quad +cx^2+c(a+b)x+abc \\
\hline
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc
\end{array}$$

Ex 7. Find the continued product of $x-a$, $x-b$ and $x-c$

$$\begin{array}{l}
x-a \\
x-b \\
\hline
x^2-xa \\
\quad -bx+ab \\
\hline
x^2-(a+b)x+ab
\end{array}
\qquad\qquad
\begin{array}{l}
x^2-(a+b)x+ab \\
x-c \\
\hline
x^3-(a+b)x^2+abx \\
\quad -cx^2 \qquad +c(a+b)x-abc \\
\hline
x^3-(a+b+c)x^2+(ab+ac+bc)x-abc
\end{array}$$

EXERCISE 8.

Multiply —

1. a^2-a+1 by a^2+a+1 2. x^2-xy+y^2 by $x+y$.
3. x^2+xy+y^2 by $x-y$
4. $x^2-2xy+y^2$ by $x^2+2xy+y^2$
5. a^4-a^2+1 by a^4+a^2+1

6. $m^4+m^2n^2+n^4$ by m^2-n^2.
7. $y^3+y^2x^2+x^3$ by $x^3-x^2y^2+y^3$.
8. $x^3+3x+3x^2+1$ by $x^3+3x-3x^2-1$.
9. $a^4-5ab+3b^2a+4a^2b-b^4$ by $a^4+3b^2a+5ab-4a^2b+b^4$.
10. $x^2-xy+y^2-yz+z^2-zx$ by $y+z+x$.
11. $b^4+b^2a^2+ba^3+b^3a+a^4$ by $a^4+b^4+b^2a^2-b^3a-ba^3$.
12. $11a-7b+4c-5d$ by $5d-4c+7b-11a$.
13. $p^4q^4-p^2q^2+p^3q^3+pq$ by $p-q$.
14. $9a^2-24ab+16b^2$ by $3a-4b$.
15. $a^3-x^3+3ax^2-3a^2x$ by a^2+x^2-2ax.
16. $1+2x+x^4+2x^3+3x^2$ by $1+x^2-2x$.

Multiply together :—

17. $a+b$, $a-b$ and a^2-b^2.
18. a^2+ab+b^2, a^2-ab+b^2 and $a^4-a^2b^2+b^4$.
19. $x+a$, $x+b$, $x+c$ and $x+d$.
20. x^2+3x+1, x^2-3x+1 and x^4+7x^2+1.
21. $a+b$, $b+c$ and $c+a$.
22. $a-b$, $b-c$ and $c-a$.
23. $a+b+c$, $b+c-a$, $c+a-b$ and $a+b-c$.
24. x^4+y^4, x^2+y^2, $x+y$ and $x-y$.
25. a^2-ab+b^2, a^2+ab+b^2, $a+b$ and $a-b$.

CHAPTER V.

MULTIPLICATION.—*Continued.*

FORMULÆ AND THEIR APPLICATION.

38. A formula is the most general expression for any theorem respecting numerical quantities.

39. The square of the sum of two quantities is equal to the sum of their squares increased by twice the product of the two quantities.

Formula $(a+b)^2 = a^2 + b^2 + 2ab$.

Ex. 1. Find the square of $3x+4y$.

$$(3x+4y)^2 = (3x)^2 + (4y)^2 + 2(3x)(4y)$$
$$= 9x^2 + 24xy + 16y^2.$$

Ex. 2. Find the square of $2a^3+5b^2$.

$$(2a^3+5b^2)^2 = (2a^3)^2 + (5b^2)^2 + 2(2a^3)(5b^2)$$
$$= 4a^6 + 20a^3b^2 + 25b^4.$$

40. The square of the difference of two quantities is equal to the sum of their squares diminished by twice the product of the two quantities.

Formula $(a-b)^2 = a^2 + b^2 - 2ab$.

Ex. 1. Find the square of $2a-3b$.

$$(2a-3b)^2 = (2a)^2 + (3b)^2 - 2(2a)(3b) = 4a^2 - 12ab + 9b^2.$$

Ex. 2. Find the square of $3a^4-4b^5$.

$$(3a^4-4b^5)^2 = (3a^4)^2 + (4b^5)^2 - 2(3a^4)(4b^5)$$
$$= 9a^8 - 24a^4b^5 + 16b^{10}.$$

41. To find the square of $a+b+c$.

$$(a+b+c)^2 = \{a+(b+c)\}^2 \text{ (considering } b+c \text{ as one term)}$$
$$= a^2 + 2a(b+c) + (b+c)^2$$
$$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$$
$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$
$$= a^2 + b^2 + c^2 + 2a(b+c) + 2b(c) \quad . \; . \; . \quad \text{(i)}$$

To find the square of $a+b+c+d$.

$$(a+b+c+d)^2 = \{(a+b)+(c+d)\}^2 \quad \left(\begin{array}{l}\text{considering } a+b \text{ as}\\ \text{one term and } c+d \text{ as}\\ \text{another}\end{array}\right)$$

$$= (a+b)^2 + 2(a+b)(c+d) + (c+d)^2$$

$$= a^2 + 2ab + b^2 + 2ac + 2ad + 2bc + 2bd + c^2 + 2cd + d^2$$

$$= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$$

$$= a^2 + b^2 + c^2 + d^2 + 2a(b+c+d) + 2b(c+d) + 2c(d) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

From (i) and (ii), it is plain that the square of the sum of any number of quantities is equal to the sum of the squares of all the quantities together with twice the product of each term and the sum of all the terms which follow it.

Formula $(a+b+c+d+e \ldots)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + \ldots\ldots + 2a(b+c+d+e \ldots) + 2b(c+d+e \ldots) + 2c(d+e \ldots) + 2d(e+ \ldots)$ *etc.*

Ex. 1. Find the square of $x-y-z$.

$$(x-y-z)^2 = \{x+(-y)+(-z)\}^2$$

$$= x^2 + (-y)^2 + (-z)^2 + 2x(-y-z) + (-2y)(-z)$$

$$= x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$$

Ex. 2. Find the square of $2x+3y+4z$.

$$(2x+3y+4z)^2 = (2x)^2 + (3y)^2 + (4z)^2 + 2(2x)(3y+4z) + 2(3y)(4z)$$

$$= 4x^2 + 9y^2 + 16z^2 + 12xy + 16xz + 24yz.$$

Note.—The rules stated in Articles 39 and 40 may be applied to deduce the squares of all multinomial expressions.

Ex 1. Find the square of $a-b-c-d$.

$$(a-b-c-d)^2 = \{(a-b)-(c+d)\}^2 \quad \left(\begin{array}{l}\text{considering } a-b \text{ as one}\\ \text{term and } c+d \text{ as another}\end{array}\right)$$

$$= (a-b)^2 - 2(a-b)(c+d) + (c+d)^2$$

$$= a^2 - 2ab + b^2 - 2(ac - bc + ad - bd) + c^2 + 2cd + d^2$$

$$= a^2 + b^2 + c^2 + d^2 - 2ab - 2ac + 2bc - 2ad + 2bd + 2cd.$$

Ex. 2. Find the square of $x-y-z$.

$$(x-y-z)^2 = \{(x-y)-z\}^2 \quad (\text{considering } x-y \text{ as one term})$$

$$= (x-y)^2 - 2(x-y)z + z^2$$

$=x^2-2xy+y^2-2xz+2yz+z^2$
$=x^2+y^2+z^2-2xy-2xz+2yz.$

42. The formulæ of Articles 39 and 40 may be applied to the simplification of algebraical expressions.

Ex. 1. Simplify $(x+y+z)^2+2(x+y+z)(x-y-z)$
$+(x-y-z)^2.$

Let $x+y+z=a$; and $x-y-z=b$; then the given expression
$=a^2+2ab+b^2=(a+b)^2.$
$=(x+y+z+x-y-z)^2=(2x)^2=4x^2.$

Ex. 2. Simplify $(ab-bc+ac)^2+(ab-bc-ac)^2$
$-2(ab-bc+ac)(ab-bc-ac).$

Putting x for $ab-bc+ac$ and y for $ab-bc-ac$, we have the given expression $=x^2+y^2-2xy$.
$=(x-y)^2=\{(ab-bc+ac)-(ab-bc-ac)\}^2$
$=(2ac)^2=4a^2c^2.$

EXERCISE 9.

Find the square of each of the following expressions :—

1. $4x+3.$
2. $11x+8.$
3. $3a+8b.$
4. $4m+6n.$
5. $ax+2by.$
6. $3abc+c^2.$
7. $ax+bcy.$
8. $mp+qn.$
9. $3a^2+7b^2.$
10. $3p^2+q^2.$
11. $ab+ac+bc.$
12. $a+3b+4c.$
13. $x^2+y^2+z^2.$
14. $2m+3n+4p.$
15. $a^2+b^3+c^4.$
16. $a+b+2x+3y.$
17. $4a+3b+2c+d.$
18. $x+2y+3z+4u.$
19. $4m+3n+2p+3q.$
20. $p^2+2q^2+3m^2+4n^2.$

Find the square of each of the following expressions :—

21. $5x-3.$
22. $9x-7.$
23. $ab-ac.$
24. $-5a-4b.$
25. $4p-3q.$
26. $-ab-cd.$
27. $b^2c-a^2d.$
28. $p^3-2qr.$
29. $-abc-de.$
30. $4a^3-5c^3.$
31. $a-3b-4c.$
32. $3x-4y-5z.$
33. $4p-3m-n.$
34. $a^2-2b^2-3c^2.$
35. $a-b-x-y.$
36. $2a-3b-4c+5y.$
37. $a^2-a-1.$
38. $1000-2.$
39. $500-7.$
40. $3000-15.$

Simplify :—

41. $(a+b)^2 \pm 2(a+b)(a-b)+(a-b)^2$.

42. $(a-b+c)^2 \pm 2(a-b+c)(b+c-a)+(b+c-a)^2$.

43. $(4a-3b)^2 \pm 2(4a-3b)(3a-4b)+(3a-4b)^2$.

44. $(ab+ac-bc)^2 \pm 2(ab+ac-bc)(bc+ab-ac)$
$+(bc+ab-ac)^2$.

45. $(2a^2-b^2-c^2)^2 \pm 2(2a^2-b^2-c^2)(2b^2-a^2-c^2)$
$+(2b^2-a^2-c^2)^2$.

Find the value of :—

46. $4a^2+28a+49$ when $a=-4$.

47. $b^2c^2-12abc+36a^2$ when $a=4$, $b=13$ and $c=2$.

48. $16p^2-24pq+9q^2$ when $p=3$ and $q=4$.

49. $25(x+y)^2+20(x+y)z+4z^2$ when $x=-2$, $y=-3$
and $z=12$

50. $9(a+b)^2-30(a+b)c+25c^2$ when $a=4$, $b=3$ and $c=6$.

43. The sum of the squares of the sum and difference of any two quantities is equal to twice the sum of the squares of the two quantities.

Formula $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

44. The difference of the squares of the sum and difference of any two quantities is equal to four times the product of the two quantities.

Formula $(a+b)^2-(a-b)^3=4ab$.

EXERCISE 10.

Simplify :—

1. $(3a+7)^2+(3a-7)^2$. 2. $(a^2+b^2)^2+(a^2-b^2)^2$.

3. $(a^2+b^2+ab)^2+(a^2+b^2-ab)^2$.

4. $(x^2-y^2+2xy)^2+(x^2-y^2-2xy)^2$.

5. $(a-b+c)^2+(a-c+b)^2$.

6. $(a^3+b^3)^2+(a^3-b^3)^2$. 7. $(3a+8)^2-(3a-8)^2$.

8. $(p^2+q^2)^2-(p^2-q^2)^2$.

9. $(x^2+y^2+xy)^2-(x^2+y^2-xy)^2$.

10. $(a^2-b^2+2ab)^2-(a^2-b^2-2ab)^2$.

11. $(x-y+z)^2-(x-z+y)^2$.

12. $(x^3+y^3)^2-(x^3-y^3)^2$.

45. The product of the sum and difference of two quantities is equal to the difference of their squares.

Formula $(a+b)(a-b)=a^2-b^2$.

N.B.—Conversely, $a^2-b^2=(a+b)(a-b)$. Therefore the factors of a^2-b^2 are $a+b$ and $a-b$.

Ex. 1. Multiply $4a+3b$ by $4a-3b$.

$(4a+3b)(4a-3b)=(4a)^2-(3b)^2=16a^2-9b^2$.

Ex. 2. Multiply a^3+b^3 by a^3-b^3.

$(a^3+b^3)(a^3-b^3)=(a^3)^2-(b^3)^2=a^6-b^6$.

Ex. 3. Multiply $a+b-c$ by $b+c-a$.

$(a+b-c)(b+c-a)=\{b+(a-c)\}\{b-(a-c)\}$
$=b^2-(a-c)^2=b^2-(a^2-2ac+c^2)$
$=b^2-a^2+2ac-c^2$.

Ex. 4. Multiply a^2-ab+b^2 by a^2+ab+b^2.

$(a^2+ab+b^2)(a^2-ab+b^2)=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$
$=(a^2+b^2)^2-(ab)^2$
$=a^4+2a^2b^2+b^4-a^2b^2$
$=a^4+a^2b^2+b^4$.

Ex 5. Simplify $(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$.

The given expression $=\{x^2+xy+y^2+x^2-xy+y^2\}$
$\times\{x^2+xy+y^2-(x^2-xy+y^2)\}$
$=(2x^2+2y^2)\times 2xy$
$=4xy(x^2+y^2)$.

Ex. 6. Find the value of $(1000)^2-(999)^2$.

The expression $=(1000+999)(1000-999)$
$=1999\times 1=1999$.

Ex. 7. Resolve into factors $(x+y)^2-(a-b)^2$.

The expression $=\{(x+y)+(a-b)\}\{(x+y)-(a-b)\}$
$=(x+y+a-b)(x+y-a+b)$.

Ex. 8. Resolve into factors $4a^2b^2-25c^4$.

The expression $=(2ab)^2-(5c^2)^2$
$=(2ab+5c^2)(2ab-5c^2)$.

Ex. 9. Resolve into factors a^8-16b^8.

The expression $=(a^4)^2-(4b^4)^2=(a^4+4b^4)(a^4-4b^4)$.

Again $a^4-4b^4=(a^2)^2-(2b^2)^2=(a^2+2b^2)(a^2-2b^2)$

$\therefore$ the given expression $=(a^4+4b^4)(a^2+2b^2)(a^2-2b^2)$.

Ex. 10. Find the value of $p^2+q^2+2pq-r^2$ when $p=2$, $q=3$ and $r=-5$.

The expression $=(p+q)^2-r^2=(p+q+r)(p+q-r)$
$=(2+3-5)(2+3+5)=0\times 10=0$

EXERCISE 11.

Multiply together :—

1. $ab+cd$ by $ab-cd$.
2. $5a+7$ by $5a-7$.
3. $ax+y^2$ by $ax-y^2$.
4. $xy+yz$ by $xy-yz$.
5. $4x+5b$ by $4x-5b$.
6. $3x^2-7b^2$ by $3x^2+7b^2$.
7. $4xy-5mn$ by $4xy+5mn$.
8. p^2-3qr by p^2+3qr.
9. a^2+b^2, $a+b$ and $a-b$.
10. $p+1$, $p-1$ and p^2+1.
11. $x+y-z$ by $z-y+x$.
12. $x+y+z$ by $x-y+z$.
13. p^2+pq+q^2 by p^2-pq+q^2.
14. a^4+a^2+1 by a^4-a^2+1.
15. $x^2+2xy+y^2$ by $x^2-2xy+y^2$.
16. $a+\sqrt{ab}+b$ by $a-\sqrt{ab}+b$.
17. a^2-2a+1, a^2+2a+1 and a^4+2a^2+1.
18. $x+\sqrt{3x}+3$, x^2-3x+9 and $x-\sqrt{3x}+3$.

Resolve into factors :—

19. $a^2-(b-c)^2$.
20. $(a+b)^2-(c+d)^2$.
21. $25a^2-9$.
22. $4a^2x^2-9b^2$.
23. $49c^2-64d^2$.
24. a^4-81b^4.
25. $(a+b-c)^2-(a-b+c)^2$.
26. $16p^2-49q^2$.
27. $x^2-(2y-3z)^2$.
28. $(4a-3c)^2-25a^2$.
29. $(3x+4y)^2-49z^2$.
30. $(2a+7b)^2-(4a-5b)^2$.
31. $(4p-5q)^2-(3p+7q)^2$.
32. $(m-3n)^2-(3n-m)^2$.
33. $(a-2b+3c)^2-(3a-2b+c)^2$.
34. $(2x+3y-4z)^2-(4x-3y+2z)^2$.
35. $(3p-4q+r)^2-(4r-3q+p)^2$.
36. $(a^2+b^2-c^2)^2-4a^2b^2$.

Find the value of :—

37. $x^2+y^2+2xy-z^2$ when $x=3$, $y=2$ and $z=5$.
38. $a^2+b^2-2ab-c^2$ when $a=\cdot 1$, $b=\cdot 3$ and $c=\cdot 2$.
39. $(49346)^2-(49336)^2$.
40. $3434\times 3434-3432\times 3432$.

46. The cube of the sum of any two quantities is equal to the sum of the cubes of the two quantities, increased by thrice their product multiplied by their sum.

Formula $(a+b)^3=a^3+b^3+3ab(a+b)$.

Ex. 1. Find the cube of $2x+3y$.

$$\begin{aligned}(2x+3y)^3&=(2x)^3+(3y)^3+3(2x)(3y)(2x+3y)\\&=8x^3+27y^3+18xy(2x+3y)\\&=8x^3+27y^3+36x^2y+54xy^2\\&=8x^3+36x^2y+54xy^2+27y^3.\end{aligned}$$

Ex. 2. Simplify $(x+y)^3+(x-y)^3+3(x-y)^2(x+y)$
$+3(x-y)(x+y)^2$.

Putting a for $x+y$ and b for $x-y$, we have the given expression $=a^3+b^3+3b^2a+3ba^2$

$$\begin{aligned}&=a^3+b^3+3ab(a+b)\\&=(a+b)^3=\{(x+y)+(x-y)\}^3\\&=(2x)^3=8x^3.\end{aligned}$$

Ex. 3. If $p+q=5$ and $pq=6$, find p^3+q^3.

$$(p+q)^3=p^3+q^3+3pq(p+q)$$

$$\therefore\ (5)^3=p^3+q^3+3\times6\times5$$

$$\therefore\ 125=p^3+q^3+90,\quad \therefore\ p^3+q^3=125-90=35.$$

Ex. 4. If $x+\frac{1}{x}=2$, find $x^3+\frac{1}{x^3}$.

$$\left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=x^3+\frac{1}{x^3}$$
$$+3\left(x+\frac{1}{x}\right)$$

$$\therefore\ 2^3=x^3+\frac{1}{x^3}+3\times2$$

$$\therefore\ 8=x^3+\frac{1}{x^3}+6,\quad \therefore\ x^3+\frac{1}{x^3}=8-6=2.$$

Similarly $x^9+\frac{1}{x^9}=2$; $x^{27}+\frac{1}{x^{27}}=2$, &c.

The cube of the sum of three or more quantities can be deduced from the formula of Art. 46.

$(a+b+c)^3=\{(a+b)+c\}^3$ (considering $a+b$ as one term)

$=(a+b)^3+c^3+3(a+b)(c)(a+b+c)$

$=a^3+b^3+3ab(a+b)+c^3+3c(a+b)^2+3c^2(a+b)$

$=a^3+b^3+c^3+3a^2b+3ab^2+3ca^2+3cb^2+6abc+3c^2a$
$+3c^2b$

$=a^3+b^3+c^3+3ab(a+b)+3bc(b+c)+3ac(a+c)$
$+6abc$. (i)

$=a^3+b^3+c^3+3a^2(b+c)+3b^2(a+c)+3c^2(a+b)$
$+6abc$. (ii)

$=a^3+b^3+c^3+3a(b^2+c^2)+3b(a^2+c^2)+3c(a^2+b^2)$
$+6abc$. (iii)

$=a^3+b^3+c^3+3(a+b+c)(ab+ac+bc)-3abc$ . . . (iv)

$=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$ . . . . (v)

Note.—These five forms are *very important.*

47. The cube of the difference of any two quantities is equal to the difference of the cubes of the two quantities, diminished by thrice their product multiplied by their difference.

Formula $(a-b)^3=a^3-b^3-3ab(a-b)$.

Ex. 1. Find the cube of $2x-3y$.

$$(2x-3y)^3=(2x)^3-(3y)^3-3(2x)(3y)(2x-3y)$$
$$=8x^3-27y^3-18xy(2x-3y)$$
$$=8x^3-27y^3-36x^2y+54xy^2$$
$$=8x^3-36x^2y+54xy^2-27y^3.$$

Ex. 2. Find the cube of $a-b-c$.

$(a-b-c)^3=\{(a-b)-c\}^3$ (considering $a-b$ as one term)

$=(a-b)^3-c^3-3(a-b)(c)(a-b-c)$

$=a^3-b^3-3ab(a-b)-c^3-3c(a-b)^2+3c^2(a-b)$

$=a^3-b^3-c^3-3a^2b+3ab^2-3ca^2-3cb^2+6abc+3c^2a$
$-3c^2b$

$=a^3-b^3-c^3-3ab(a-b)+3bc(-b-c)-3ac(a-c)+6abc.$

Note.—The same may be deduced from the expansion of $(a+b+c)^3$ by putting $-b$ for b and $-c$ for c.

EXERCISE 12.

Find the cube of :—

1. $4a+3$.
2. $3x+2y$.
3. $4b-5a$.
4. $ab-cd$.
5. $2p-5q$.
6. x^2+2y.
7. $xy+yz+zx$.
8. $ab-bc+ac$.
9. $a^2-b^2-c^2$.
10. $a+2b-c$.
11. $p^2+q^2-r^2$.
12. $a+2b+3c$.

Find the value of :—

13. x^3+y^3 when $x+y=7$ and $xy=6$.
14. x^3-y^3 when $x-y=6$ and $xy=16$
15. $a^3+\dfrac{1}{a^3}$ when $a+\dfrac{1}{a}=3$.
16. $a^3-\dfrac{1}{a^3}$ when $a-\dfrac{1}{a}=5$.
17. x^3+y^3+15xy when $x+y=5$
18. p^3+q^3+6pq when $p+q=2$.
19. x^3-y^3-9xy when $x-y=3$.
20. $8x^3-27y^3-90xy$ when $2x-3y=5$.

Simplify :—

21. $(a+b+c)^3-(a^3+b^3+c^3)$.
22. $(x-y)^3+(y-z)^3+(z-x)^3$
23. $(2a-3b)^3-(2b-3a)^3-15(2a-3b)(2b-3a)(a-b)$.
24. $x^6+3x^4y^2+3x^2y^4+y^6$ when $x=a+b$ and $y=a-b$.
25. $(a+b+c)^3+(c-a-b)^3+6c\{c^2-(a+b)^2\}$.

48. By actual multiplication, we can establish the following Formulæ :—

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$
$$\text{and } (a-b)(a^2+ab+b^2)=a^3-b^3.$$

Note.—Conversely, the factors of a^3+b^3 are $a+b$ and a^2-ab+b^2 and those of a^3-b^3 are $a-b$ and a^2+ab+b^2.

Ex. 1. Multiply a^4-a^2+1 by a^2+1.

Putting p for a^2 and q for 1, we have

$$a^4-a^2+1=(a^2)^2-a^2\times 1+1^2=p^2-pq+q^2$$
$$\therefore (a^4-a^2+1)(a^2+1)=(p^2-pq+q^2)(p+q)=p^3+q^3$$
$$=(a^2)^3+1^3=a^6+1.$$

Ex. 2. Multiply $9a^2+12a+16$ by $3a-4$.

$$9a^2+12a+16=(3a)^2+3a\times 4+4^2$$
$$=(x^2+xy+y^2) \text{ (putting } x \text{ for } 3a \text{ \& } y \text{ for } 4)$$
$$\therefore (9a^2+12a+16)(3a-4)=(x^2+xy+y^2)(x-y)=x^3-y^3$$
$$=(3a)^3-4^3=27a^3-64.$$

Ex. 3. Resolve into factors a^3+27.

$$a^3+27=a^3+3^3=(a+3)\{a^2-a\times 3+3^2\}$$
$$=(a+3)(a^2-3a+9).$$

Ex. 4. Resolve into factors x^3-125a^3.

$$x^3-125a^3=x^3-(5a)^3=(x-5a)\{x^2+x\times 5a+(5a)^2\}$$
$$=(x-5a)(x^2+5ax+25a^2).$$

EXERCISE 13.

Multiply :—

1. x^2-2x+4 by $x+2$. 2. x^2-3x+9 by $x+3$.
3. $4a^2-2a+1$ by $2a+1$. 4. $a^2b^2-abc+c^2$ by $ab+c$.
5. $25x^2-10xy+4y^2$ by $5x+2y$.
6. $16a^2-28ab+49b^2$ by $4a+7b$.
7. $1+3x+9x^2$ by $1-3x$.
8. $a^2+2abc+4b^2c^2$ by $a-2bc$.
9. x^2+2x+4 by $x-2$.
10. $m^2+4mnp+16p^2n^2$ by $m-4pn$.
11. $4a^2b^4+2ab^2+1$ by $2ab^2-1$.
12. $a^8+a^4b^4+b^8$ by a^4-b^4.

Resolve into factors :—

13. x^3+8. 14. p^3+64. 15. $27b^3c^3+a^3$.
16. $8a^3+343b^3$. 17. $125x^3-1$. 18. $1-343x^3$.
19. $216b^3c^3-a^3$. 20. $64a^3-27b^3d^3$.

49. When two or more binomial factors have one term common to them all, their product can be readily written down.

I. Formula $(x+a)(x+b)=x^2+(a+b)x+ab$.

The following may be deduced from the above formula:—

i. $(x-a)(x-b)=x^2-(a+b)x+ab$.

ii. $(x+a)(x-b)=x^2+(a-b)x-ab$.

iii. $(x-a)(x+b)=x^2+(b-a)x-ab$.

Ex. 1. Multiply $x+7$ by $x+11$

$$(x+7)(x+11)=x^2+(7+11)x+7\times 11=x^2+18x+77.$$

Ex. 2. Multiply $x-7$ by $x+10$.

$$(x-7)(x+10)=x^2+(-7+10)x+(-7)(10)$$
$$=x^2+3x-70.$$

II Formula $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2$
$+(ab+ac+bc)x+abc.$

Similarly $(x+a)(x+b)(x+c)(x+d)=x^4$
$+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$
$+(abc+abd+acd+bcd)x+abcd.$

These results can be got by actual multiplication.

We observe the following laws in Formulæ I and II :—

1. The number of terms in the product is one more than the number of binomial factors

2. The index of the highest power of x is the same as the number of factors, and the indices of x decrease by unity in each succeeding term.

3. The co-efficient of the highest power of x is unity, and the co-efficients of the succeeding powers of x are the sum, the sum of the products in pairs, in threes, &c., respectively, of the second terms of the factors, and the last term is the product of all the second terms of the factors.

Ex 1 Multiply together $x+1$, $x+2$ and $x+3$.

$(x+1)(x+2)(x+3)=x^3+(1+2+3)x^2+(1.2+1.3+2.3)x$
$+1.2.3=x^3+6x^2+11x+6$

Ex. 2. $(x-2)(x-3)(x-4)$

$(x-2)(x-3)(x-4)=x^3+(-2-3-4)x^2$
$+\{(-2)\times(-3)+(-3)\times(-4)+(-2)\times(-4)\}x$
$+(-2)\times(-3)\times(-4)=x^3-9x^2+26x-24$

Ex 3 $(x+1)(x-2)(x-4)$

$(x+1)(x-2)(x-4)=x^3+(1-2-4)x^2$
$+\{1\times(-2)+1\times(-4)+(-2)\times(-4)\}x$
$+1\times(-2)\times(-4)=x^3-5x^2+2x+8$

Ex. 4. $(a+b)(b+c)(c+a)$.

$(a+b)(b+c)(c+a)=\{(a+b+c)-c\}\{(a+b+c)-a\}$
$\{(a+b+c)-b\}$
$=(a+b+c)^3+(-c-a-b)(a+b+c)^2+\{(-c)(-a)$
$+(-c)(-b)+(-a)(-b)\}(a+b+c)+(-c)(-a)(-b)$
$=(a+b+c)^3-(a+b+c)^3+(ac+bc+ab)(a+b+c)-abc$
$=(ab+ac+bc)(a+b+c)-abc.$

EXERCISE 14.

Write down the product of :—

1. $(x+7)(x+5)$.
2. $(x+2)(x-3)$.
3. $(x^2-3)(x^2-5)$.
4. $(x-3y)(x+4y)$.
5. $(2x-3)(2x+5)$.
6. $(ax+12)(ax-13)$.
7. $(x+2)(x+3)(x+4)$.
8. $(x-5)(x+6)(x-7)$.
9. $(2x-1)(2x-3)(2x+4)$.
10. $(x-1)(x+1)(x+3)(x-3)$.
11. $(x+\frac{1}{2})(x+\frac{1}{4})(x+\frac{1}{3})$.
12. $(1+a)(1+b)(1+c)(1+d)$.
13. $(b+c-a)(c+a-b)(a+b-c)$.
14. $(2a+b+c)(2b+c+a)(2c+a+b)$.
15. $(x+a-b)(x+b-c)(x+c-a)$.
16. $(y+z-2x)(z+x-2y)(x+y-2z)$.

50. By actual multiplication we can get the following :—

Formula. $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
$$=a^3+b^3+c^3-3abc.$$

Note.—Conversely, the factors of $a^3+b^3+c^3-3abc$ are $(a+b+c)$ and $(a^2+b^2+c^2-ab-ac-bc)$.

Ex. 1. Multiply $x^2+y^2+z^2-xy+xz+yz$ by $x+y-z$.

Putting p for x, q for y and r for $-z$, we have

$$(x+y-z)(x^2+y^2+z^2-xy+xz+yz)$$
$$=(p+q+r)(p^2+q^2+r^2-pq-pr-qr)=p^3+q^3+r^3-3pqr$$
$$=x^3+y^3+(-z)^3-3xy(-z)=x^3+y^3-z^3+3xyz.$$

Ex. 2. Multiply $a^2+4b^2+9c^2-2ab-3ac-6bc$ by $a+2b+3c$.

Putting p for a, q for $2b$ and r for $3c$, we have

$$(a+2b+3c)(a^2+4b^2+9c^2-2ab-3ac-6bc)$$
$$=(p+q+r)(p^2+q^2+r^2-pq-pr-qr)=p^3+q^3+r^3-3pqr$$
$$=a^3+(2b)^3+(3c)^3-3a(2b)(3c)=a^3+8b^3+27c^3-18abc.$$

EXERCISE 15.

Multiply :—

1. $x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2$ by $x^2+y^2+z^2$.
2. $a^2+b^2+c^2+ab+ac-bc$ by $a-b-c$.
3. $4x^2+9y^2+z^2+6xy+2xz-3yz$ by $2x-3y-z$.

4. $9p^2+25q^2+16+15pq+12p-20q$ by $3p-5q-4$.

5. $m^2+n^2+1+mn-m+n$ by $m-n+1$.

7. $x^2+\frac{1}{x^2}-x-\frac{1}{x}$ by $x+\frac{1}{x}+1$.

8. $(a+b)^2+(b+c)^2+(c+a)^2-(a+b)(b+c)$
$-(b+c)(c+a)-(c+a)(a+b)$ by $2(a+b+c)$.

7. $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{c}-\frac{b}{a}-\frac{c}{b}$ by $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

51. Formula. $(a+b)(b+c)(c+a)$

$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc \quad \ldots \quad \text{(i)}$$
$$=a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc \quad \ldots \quad \text{(ii)}$$
$$=a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc \quad \ldots \quad \text{(iii)}$$
$$=a^2(b+c)+b^2(c+a)+c^2(b+a)+2abc \quad \ldots \quad \text{(iv)}$$
$$=ab(a+b)+bc(b+c)+ac(a+c)+2abc \quad \ldots \quad \text{(v)}$$
$$=(a+b+c)(ab+ac+bc)-abc \quad \ldots \quad \text{(vi)}$$

Note.—Conversely, the factors of an expression of any one of the six forms are $(a+b)$, $(b+c)$ and $(c+a)$.

Ex. 1. Multiply $(a+2b+c)(b+2c+a)(c+2a+b)$.

Putting p for $a+b$, q for $b+c$ and r for $c+a$, we have

$$(a+2b+c)(b+2c+a)(c+2a+b)=(p+q)(q+r)(p+r)$$
$$=p(q^2+r^2)+q(p^2+r^2)+r(p^2+q^2)+2pqr$$
$$=(a+b)\{(b+c)^2+(c+a)^2\}+(b+c)\{(a+b)^2+(c+a)^2\}$$
$$+(c+a)\{(a+b)^2+(b+c)^2\}+2(a+b)(b+c)(c+a).$$

This can be expanded by Formulæ.

Ex. 2. Multiply $(x+y+z)$ by $(xy+yz+zx)$.

The product $=(x+y)(y+z)(z+x)+xyz$
$$=x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+3xyz.$$

52. Formula. $(a-b)(b-c)(c-a)$

$$=a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) \quad \ldots \quad \text{(i)}$$
$$=a^2(c-b)+b^2(a-c)+c^2(b-a) \quad \ldots \quad \text{(ii)}$$
$$=ab(b-a)+bc(c-b)+ca(a-c) \quad \ldots \quad \text{(iii)}$$

Note. Conversely, the factors of an expression of any one of the three forms are $a-b$, $b-c$ and $c-a$.

Ex Multiply $(x^2-y^2)(y^2-z^2)(z^2-x^2)$.

The given expression $=(a-b)(b-c)(c-a)$, putting a for x^2, b for y^2 and c for z^2

$$=\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}$$
$$=\{x^2(y^4-z^4)+y^2(z^4-x^4)+z^2(x^4-y^4)\}$$
$$=x^2y^4-x^2z^4+y^2z^4-y^2x^4+x^4z^2-y^4z^2.$$

53. Formula. $(a+b+c)(a^2+b^2+c^2)$
$$=a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b).$$

$$(a+b+c)(a^2+b^2+c^2)=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$
$$+(a+b+c)(ab+ac+bc)$$
$$=a^3+b^3+c^3-3abc+a^2(b+c)+b^2(c+a)$$
$$+c^2(a+b)+3abc$$
$$=a^3+b^3+c^3\ +a^2(b+c)+b^2(c+a)+c^2(a+b).$$

Note.—Conversely, the factors of $a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)$ are $a+b+c$ and $a^2+b^2+c^2$.

Ex. Multiply $(x+y+2)$ by (x^2+y^2+4).

The product $=x^3+y^3+2^3\ +x^2(y+2)+y^2(x+2)+2^2(x+y)$
$$=x^3+y^3+8+x^2y+y^2x+2x^2+2y^2+4x+4y.$$

EXERCISE 16.

Simplify by formulæ:—

1. $(1+a+b)(a+b+ab)$. 2. $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)$.

3. $2(x+y+z)\{(x+y)(x+z)+(x+z)(z+y)+(z+y)(x+y)\}$.

4. $(x-2y+z)(y-2x+z)(x-2z+y)$.

5. $(a+b)^2(b-a)+(b+c)^2(c-b)+(c+a)^2(a-c)$
$+(b-a)(c-b)(a-c)$.

6. $(a+2b+3c)^2(a-2b+c)+(b+2c+3a)^2(b-2c+a)$
$+(c+2a+3b)^2(c-2a+b)+(a-2b+c)(b-2c+a)$
$\times(c-2a+b)$.

7. $(1+x)(x+y)(y+1)$. 8. $(2b+c)(b+c)(2c+b)$.

9. $(2a+2b+c)(b+c)(2c+2a+b)$.

10. $(x^2+4y^2+9c^2)(x-2y-3c)$.

11. $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{a}+\frac{a}{b}\right)$

12. $(a-b)(b-c)(a-2b+c)+(b-c)(c-a)(b-2c+a)$
$+(c-a)(a-b)(c-2a+b)$

CHAPTER VI.

DIVISION.

54. Division is the inverse of Multiplication. In Multiplication we determine the product of two given factors ; in Division we have the product and one of the factors given, and our object is to determine the other factor.

When one quantity is divided by another, the former is called the *dividend*, and the latter the *divisor* ; the result is called the *quotient*.

55. Rule of Signs.—" When the dividend and the divisor have the same sign the quotient is positive, and when they have different signs the quotient is negative. More briefly, " like signs produce + and unlike signs —."

Thus $\frac{+ab}{+a} = +b$; $\frac{-ab}{-a} = +b$; $\frac{+ab}{-a} = -b$; $\frac{-ab}{+a} = -b$.

56. There are three cases in Division :—

First Case. When the divisor and dividend are *both simple quantities* or *monomials*.

Rule.—" Write the divisor under the dividend in the form of a fraction and divide the two terms of the fraction by all the factors which are common to both and prefix to the result the proper sign."

Ex. Divide $16m^2np$ by $-6mn^2p$.

The dividend $= 2 \times 8 \times m \times m \times n \times p$.

The divisor $= -2 \times 3 \times m \times n \times n \times p$.

$\therefore$ the quotient $= \frac{8 \times m}{-3 \times n} = -\frac{8m}{3n}$.

When the dividend and divisor are powers of the same letter or quantity, we subtract the index of the power in the divisor from the index of the power in the dividend, or *vice versâ*, according as the former index is less or greater than the latter.

Thus $\frac{a^5}{a^3} = a^{5-3} = a^2$ and $\frac{a^2}{a^4} = \frac{1}{a^{4-2}} = \frac{1}{a^2}$.

Generally, when m and n are positive integers,

$$\frac{a^m}{a^n} = a^{m-n} \text{ when } m > n.$$

For $\frac{a^m}{a^n}=\frac{a\times a\times a\ \ldots\ \text{to } m \text{ factors}}{a\times a\times a\ \ldots\ \text{to } n \text{ factors}}=a\times a\times a\ \ldots$ to $(m-n)$ factors $=a^{m-n}$.

Also $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$, **when $n>m$.**

For $\frac{a^m}{a^n}=\frac{a\times a\times a\ \ldots\ \text{to } m \text{ factors}}{a\times a\times a\ \ldots\ \text{to } n \text{ factors}}=\frac{1}{a\times a\times a \text{ to } (n-m) \text{ factors}}$
$=\frac{1}{a^{n-m}}$.

To prove that $a^0=1$.

In $\frac{a^m}{a^n}=a^{m-n}$, let $m=n$. Then $\frac{a^n}{a^n}=a^{n-n}$; or $1=a^0$.

Note. – Any expression raised to the power of 0 is $=1$.

EXERCISE 17.

Divide :—

1. ax by $-a$. 2. abx by $-b$. 3. $a^3b^3x^2$ by $-3abx$.
4. $-14a^3b^2c^3$ by $28a^4b^3c^2$. 5. $a^3(b-c)^2$ by $a^2(b-c)^3$.
6. ax^my^n by $bx^{m+1}y^{n+2}$. 7. $x^{m+p}y^n$ by x^py^{n+p}.
8. $x^{m-n}y^{m+n}z^{2n}$ by $x^{m+n}y^{m-n}z^n$. 9. $x^{a-b}\times x^{2b+2a}$ by x^{3a+b}.

57. *Second Case.* When the *divisor alone is a simple quantity.*

Rule.—" Divide each term of the dividend by the divisor and add the quotients."

Ex. 1. Divide $ab+ac$ by a.

$$\text{Quotient}=\frac{ab}{a}+\frac{ac}{a}=b+c.$$

Ex. 2. Divide $14x^2y-8x^3z+6z^3$ by $2xyz$.

$$\text{Quotient}=\frac{14x^2y}{2xyz}-\frac{8x^3z}{2xyz}+\frac{6z^3}{2xyz}$$
$$=\frac{7x}{z}-\frac{4x^2}{y}+\frac{3z^2}{xy}.$$

58. *Third Case.* When both the divisor and the dividend contain *more than one term.*

Rule.—" Arrange the divisor and the dividend according to descending or ascending powers of some common letter, divide the first term of the dividend by the first term of the divisor, the quotient obtained is the first term of the required

quotient; multiply the divisor by it, and subtract the product from the dividend (bringing down only as many terms of the dividend as are necessary); repeat the operation, using each new remainder as dividend, till all the terms in the dividend are exhausted and the remainder vanishes; or if the remainder does not vanish till we have as many terms in the quotient as we require, then set down as the last term of the quotient the remainder over the divisor in the form of a fraction."

Ex. 1. Divide $1+3x^2+x^3+3x$ by $x+1$. We first arrange according to descending powers of a, thus :—

Divisor Dividend. Quotient.

$$x+1)x^3+3x^2+3x+1(x^2+2x+1$$
$$x^3+x^2$$
$$2x^2+3x$$
$$2x^2+2x$$
$$x+1$$
$$x+1$$

Ex 2. Divide x^4+64 by x^2+4x+8.

$$x^2+4x+8)x^4+64\qquad(x^2-4x+8$$
$$x^4+4x^3+8x^2$$
$$-4x^3-8x^2+64$$
$$-4x^3-16x^2-32x$$
$$8x^2+32x+64$$
$$8x^2+32x+64$$

Ex. 3. Divide $x^3-(a+b+c)x^2+(bc+ca+ab)x$ $-abc$ by $x^2-(b+c)x+bc$.

$$x^2-(b+c)x+bc)x^3-(a+b+c)x^2+(bc+ca+ab)x-abc(x-a$$
$$x^3-(b+c)x^2+bcx$$
$$-ax^2+(ca+ab)x-abc$$
$$-ax^2+(ab+ac)x-abc$$

Ex. 4. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$. We first arrange according to descending powers of a, thus :—

$$a+(b+c))a^3-3abc+(b^3+c^3)(a^2-(b+c)a+(b^2-bc+c^2)$$
$$a^3+a^2(b+c)$$
$$-a^2(b+c)-3abc+(b^3+c^3)$$
$$-a^2(b+c)-(b+c)^2a$$
$$(b^2-bc+c^2)a+(b^3+c^3)$$
$$(b^2-bc+c^2)a+(b^3+c^3)$$

Note.—The quotient may be written from the formula $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$.

Ex. 5. Divide $16x^4+36x^2+6x+86$ by $4x^2+6x+9$.

$$
\begin{array}{l}
4x^2+6x+9)\,16x^4+36x^2+6x+86\,(4x^2-6x+9 \\
\qquad\qquad\quad \underline{16x^4+24x^3+36x^2} \\
\qquad\qquad\qquad\quad -24x^3+6x+86 \\
\qquad\qquad\qquad\quad \underline{-24x^3-36x^2-54x} \\
\qquad\qquad\qquad\qquad\quad 36x^2+60x+86 \\
\qquad\qquad\qquad\qquad\quad \underline{36x^2+54x+81} \\
\qquad\qquad\qquad\qquad\qquad\quad 6x+5\;.\;.\;\textit{Remainder.}
\end{array}
$$

Thus the quotient is $4x^2-6x+9+\dfrac{6x+5}{4x^2+6x+9}$.

Note.—If in dividing M by N, we have Q for the quotient and R for the remainder, then $\dfrac{M}{N}=Q+\dfrac{R}{N}$, *i.e.*, $M=NQ+R$.

EXERCISE 18.

Divide :—

1. x^3+2x^2+3x by x.
2. $x^7-x^3+2x^2$ by $-x^2$.
3. $a^4x^4-a^6x^4$ by a^4x^4.
4. $abc^2+ab^2c+a^2bc$ by abc.
5. $9a^2b+27a^3b^2-81ab^3$ by $-27a^2b^2$.
6. $x^{m+2}+x^{m+1}+x^m$ by x^m.
7. x^2+3x+2 by $x+1$.
8. x^2-7+10 by $x-5$.
9. $6x^2-13x+6$ by $2x-3$.
10. x^3-3x^2+3x-1 by x^2-2x+1.
11. $3x^3-10x^2+x+6$ by x^2-4x+3.
12. x^6-a^6 by $x^3-2ax^2+2a^2x-a^3$.
13. $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.
14. x^4+64 by x^2-4x+8.
15. x^4-256 by $x^3+4x^2+16x+64$.
16. x^7+2187 by $x+3$.
17. $x^3-13x-30$ by x^2-2x+3.
18. $1-5x^4+4x^5$ by $(1-x)^2$.
19. $3x^5-10x^4y+16x^3y^2-12x^2y^3+xy^4+2y^5$ by $(x-y)^2$.
20. $21x^5-2x^4-70x^3-23x^2+33x+27$ by $7x^2+4x-9$.
21. $1+x+x^2+x^3+x^4+x^6+x^7+x^8+x^9+x^{15}$ by $1-x^5+x^6$.
22. $x^3-2ax^2+(a^2-ab-b^2)x+a^2b+ab^2$ by $x-a-b$.
23. $x(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$.
24. $(x^3-y^3)a^3-(x^3-x^2+y^2-y^3)a^2-(4x^2+3xy+2y^2)a-3(x+y)$ by $(x-y)(a^2-a)-3$.
25. $3(c-b)a^2-(c^2-b^2)a+c^2(c-3b)+b^2(3c-b)$ by $3a^2-(b+c)a+(b-c)^2$.

26. $a^2(b+c)+b^2(a+c)+c^2(a+b)+3abc$ by $a+b+c$.
27. $a^2(b+c)-b^2(a+c)+c^2(a+b)+abc$ by $a-b+c$.
28. $a^3+b^3+c^3+a^2(b+c)+b^2(a+c)+c^2(a+b)$ by $a^2+b^2+c^2$.
29. $x^3+y^3-1+3xy$ by $x+y-1$.
30. $a^3-8b^3-27c^3-18abc$ by $a-2b-3c$.
31 $x^3-y^3+z^3+3xyz$ by $x-y+z$.
32. $8a^3-27b^3-c^3-18abc$ by $4a^2+9b^2+c^2+6ab+2ac-3bc$.
33. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$ by $a+b$.
34. $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc$ by $b+c$.
35. $ab(a+b)+bc(b+c)+ca(c+a)+2abc$ by $c+a$.
36. $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc$ by $ab+ac+bc$.
37. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a-b$.
38. $ab(a-b)+bc(b-c)+ca(c-a)$ by $b-c$.
39. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ by $c-a$.
40. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$.
41. $a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)$ by a^2b-bc^2 $-ac^2+a^2c$.
42. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ by $ab+ac+bc$.
43. $(ay-bx)^2+(bz-cy)^2+(cx-az)^2+(ax+by+cz)^2$ by $a^2+b^2+c^2$ and by $x^2+y^2+z^2$.
44. $4a^2b^2-(a^2+b^2-c^2)^2$ by the product of $a+b+c$ and $a+b-c$.
45. $(a-b)x^3+(b^3-a^3)x+ab(a^2-b^2)$ by $(a-b)x+a^2-b^2$.
46. $x^6-(a^2+b^2+1)x^4+(a^2b^2+a^2+b^2)x^2-a^2b^2$ by $x^2-(a-b)x-ab$.
47. $(b^2-c^2)x^4+2c(b-a)x^3+(c-a)(c+a-2b)x^2$ $+2a(c-b)x+a^2-b^2$ by $(b-c)x^2+(c-a)x+a-b$
48. $a^6(a^2-15b^2)+5a^4b^4(17a^2-45b^2)+2b^8(137a^2-60b^2)$ by $a^6-12a^4b^2+47a^2b^4-60b^6$.
49. $ay^5+(a+b)y^4+(a+b+c)y^3+(a+b+c)y^2+(b+c)y+c$ by ay^2+by+c.
50. $x^3+\frac{1}{x^3}+3\left(x^2-\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)$ by $x+\frac{1}{x}$.
51. $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$.

CHAPTER VII.

DIVISION.—*Continued.*

59. Remainder Theorem. When an algebraical expression in x is divided by $x-a$, the remainder is obtained by substituting a for x in the expression.

Dividing px^2+qx+r by $x-a$, we have

$$
\begin{array}{l}
x-a)\,px^2+qx+r\,(px+ap+q \\
\quad\;\; \underline{px^2-apx} \\
\qquad\quad x(ap+q)+r \\
\qquad\quad \underline{x(ap+q)-a(ap+q)} \\
\qquad\qquad pa^2+qa+r
\end{array}
$$

The remainder is the same as the given expression with a written instead of x.

If we divide $x^4+px^3+qx^2+rx+s$ by $x-a$, the remainder will be $a^4+pa^3+qa^2+ra+s$.

Note.—When an expression in x is divided by $x+a$, the remainder is obtained by substituting $-a$ for x in the expression. [$\because$ $x+a=x-(-a)$]. If px^2+px+r be divided by $x+a$, the remainder will be pa^2-qa+r.

If we divide $px^5+qx^4+rx^3+sx^2+tx+u$ by $x+a$, the remainder will be $-pa^5+qa^4-ra^3+sa^2-ta+u$.

60. In order that one expression may be exactly divisible by another, the remainder arising from the division must $=0$. Hence, the rule for finding the condition of perfect divisibility is "*divide as far as possible in the ordinary way, and then put the remainder*$=0$, *and deduce the necessary condition from this.*"

Ex. What value of x will make $x^3-5x^2+4x-16$ exactly divisible by x^2-3x-7?

Dividing in the usual way we get for the quotient $x-2$ and for the remainder $5x-30$. For perfect divisibility, this remainder $5x-30$ must$=0$. $\therefore$ $5x=30$; or $x=6$.

61. An expression such as $px^4+px^3+rx^2+sx+t$ will be exactly divisible by $x-a$ if the remainder arising from the division, viz., $pa^4+qa^3+ra^2+sa+t=0$; and by $x+a$ if the remainder, viz., $pa^4-qa^3+ra^2-sa+t=0$.

In other words, an expression in x is exactly divisible by $x-a$, if it vanishes when a is substituted for x; and by $x+a$ if it vanishes when $-a$ is substituted for x.

Ex. 1. x^2-a^2 vanishes when a is put for x and also when $(-a)$ is put for x; therefore x^2-a^2 is divisible by $x-a$ and $x+a$.

Ex. 2. x^3-a^3 vanishes when a is put for x; therefore it is divisible by $x-a$. When $(-a)$ is put for x, it does not vanish; therefore it is *not* divisible by $x+a$.

Ex. 3. To prove that x^n-a^n is divisible by $x-a$. Since x^n-a^n vanishes when a is put for x, $\therefore$ x^n-a^n is divisible by $x-a$.

Ex. 4. To prove that x^n+a^n is divisible by $x+a$ when n is an odd integer.

Since x^n+a^n vanishes when we write $(-a)$ for x, $\therefore$ it is divisible by $x+a$ . . $[x^n+a^n=(-a)^n+a^n=-a^n+a^n=0.$

$(-a)^n=-a^n$. $\because$ n is odd$]$

Ex. 5. To prove that x^n-a^n is divisible by $x+a$ when n is an even integer.

Since x^n-a^n vanishes when we write $(-a)$ for x, $\therefore$ it is divisible by $x+a$. $[x^n-a^n=(-a)^n-a^n=a^n-a^n=0.$ $\because$ n is even$]$.

Ex. 6. To prove that x^n-a^n is ***not*** divisible by $x+a$ when n is odd.

The expression x^n-a^n does not vanish when we write $(-a)$ for x, therefore it is not divisible by $x+a$. $[x^n-a^n=-2a^n]$.

In the same way it can be shown that x^n+a^n is *not* divisible by $x+a$ or by $x-a$, when n is even

The results arrived at in Examples 3, 4, 5 and 6 may be stated thus :—

x^n-a^n is divisible by $x-a$, when n is odd or even.

x^n-a^n is divisible by $x+a$, when n is even.

x^n-a^n is *not* divisible by $x+a$, when n is odd.

x^n+a^n is divisible by $x+a$, when n is odd.

x^n+a^n is *not* divisible by $x+a$, or $x-a$, when n is even.

62 I. By actual division, $\dfrac{x^2-a^2}{x-a}=x+a$; $\dfrac{x^3-a^3}{x-a}$

$$=x^2+xa+a^2\ ;\ \frac{x^4-a^4}{x-a}=x^3+x^2a+xa^2+a^3\ ;\ \&c.$$

We observe that the number of terms in the quotient is the same as the index of the power of x or a in the dividend; that the terms are *all positive*; and that while the indices of the powers of x decrease by unity, those of the powers of a increase by unity in each succeeding term.

Therefore $\dfrac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots + x^2a^{n-3}$

$+ xa^{n-2} + a^{n-1}.$

II. By actual division, $\dfrac{x^3 + a^3}{x+a} = x^2 - xa + a^2$; $\dfrac{x^5 + a^5}{x+a}$

$= x^4 - x^3a + x^2a^2 - xa^3 + a^4$; &c.

We observe that the terms in the quotient are alternately positive and negative; and that in other respects they obey the same laws as those of I.

Therefore $\dfrac{x^n + a^n}{x+a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots + x^2a^{n-3}$

$- xa^{n-2} + a^{n-1}.$

III. By actual division, $\dfrac{x^2 - a^2}{x+a} = x - a$;

$$\frac{x^4 - a^4}{x+a} = x^3 - x^2a + xa^2 - a^3;$$

$$\frac{x^6 - a^6}{x+a} = x^5 - x^4a + x^3a^2 - x^2a^3 + xa^4 - a^5; \text{ \&c.}$$

Here also we observe the same laws as in II.

Therefore $\dfrac{x^n - a^n}{x+a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \ldots\ldots\ldots$

$- x^2a^{n-3} + xa^{n-2} - a^{n-1}.$

Ex. 1. Write down the terms in the quotient of—

$$\frac{x^{21} + y^{21}}{x^7 + y^7}.$$

$$\frac{x^{21} + y^{21}}{x^7 + y^7} = \frac{(x^7)^3 + (y^7)^3}{x^7 + y^7} = (x^7)^2 - x^7y^7 + (y^7)^2$$

$$= x^{14} - x^7y^7 + y^{14}.$$

Ex. 2. Shew that $(a^2+bc)^n-a^n(b+c)^n$ is divided by $(a-b)(a-c)$.

Let $a^2+bc=x$ and $a(b+c)=y$; then

$x-y=a^2+bc-a(b+c)=a^2+bc-ab-ac=(a-b)(a-c)$

and $(a^2+bc)^n-a^n(b+c)^n=(a^2+bc)^n-\{a(b+c)\}^n=x^n-y^n$

$\therefore$ It is divisible by $x-y$, *i.e.*, by $(a-b)(a-c)$.

Ex. 3. If $a^{2n+1}+b^{2n+1}$ be divisible by a^3+b^3, shew that $n-1$ is a multiple of 3.

$a^{2n+1}+b^{2n+1}=(a^3)^{\frac{2n+1}{3}}+(b^3)^{\frac{2n+1}{3}}$. Since this is divisible by a^3+b^3, $\frac{2n+1}{3}$ is an odd number

$\therefore \frac{2n+1}{3}=2p+1$ (general form of an odd number) where p has values 0, 1, 2, 3, &c.

$\therefore 2n+1=6p+3$. $\therefore 2n-2=6p$. $\therefore n-1=3p$.

Ex. 4. 5^n-4n-1 is divisible by 4^2 or 16.

$5^n-4n-1=(5^n-1)-4n=(5-1)(5^{n-1}+5^{n-2}+\quad\ldots$
$\quad+5+1)-4n$

$=4\{5^{n-1}+5^{n-2}+5^{n-3}+\quad\ldots\quad+5+1-n\}$
$=4\{(5^{n-1}-1)+(5^{n-2}-1)+(5^{n-3}-1)+(5-1)+(1-1)\}$.

Splitting n into n ones.

$=4(5-1)\{(5^{n-2}+\quad\ldots\ldots\quad+1)+(5^{n-3}+\quad\ldots\ldots\quad+1)$
$\quad+\text{\&c.}\ \ldots\ldots\ +1\}$
$=4^2$ or $16\{\qquad$ do $\qquad\}$. Since 4^2 or 16 is a factor of the given expression, it is divisible by 4^2 or 16.

EXERCISE 19.

Find the remainder in each of the following without actual division :—

1. ax^3+bx^2+cx when divided by $x-1$.
2. ax^3+bx^2+cx+d by $x+p$.
3. px^5+qx^4+rx+4 by $x+a$ and by $x-a$.
4. $ax^4+bx^3+cx^2+dx+e$ by $x-2$ and by $x+2$.
5. $ma^5+na^4+pa^3+qa^2+ra+s$ by $a-b$ and by $a+b$.
6. $3y^5-2y^4+3y^3+4y^2-6y+3$ by $y+1$ and $y-1$.

What value of x will make each of the following expressions divisible by the divisor opposite to it?—

7. x^6-2x^3+x-3 by x^2-2x+1.

8. $x^3-5x^2+4x+34$ by x^2-3x-7.

9. $x^6-2x^5-5x^4+20x^3-33x^2+20x+15$ by x^2-2x+3.

10. $x^3-3x^2a+5xa^2-9a^3$ by $x-4a$ and by $x-2a$.

11. Find c so that x^2+7x+c may be divisible by $x+4$.

Shew that each of the following is divisible by the divisor opposite to it without remainder.

12. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a-b$, $b-c$ and $c-a$.

13. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$ by $a+b$, $b+c$ and $c+a$.

14. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ by $a-b$.

15. $(a+b+c)^3-a^3-b^3-c^3$ by $a+b$, $b+c$ and $c+a$.

16. x^3-3x+2 by $x-1$.

17. x^3+3x+4 by $x+1$.

18. $x^{2n}-y^{2n}$ by $x+y$ and $x-y$.

19. $a^n(b-c)+b^n(c-a)+c^n(a-b)$ by $a-b$, $b-c$ and $c-a$.

20. $(x+y+z)^5-x^5-y^5-z^5$ by $x+y$, $y+z$, and $z+x$.

Write down the quotients of :—

21. x^8-y^8 by x^2-y^2.

22. $x^{15}+y^{15}$ by x^3+y^3.

23. $x^{14}+y^{14}$ by x^2+y^2.

24. $x^{111}+y^{111}$ by $x^{37}+y^{37}$.

25. $x^{20}-y^{20}$ by x^2+y^2.

26. $x^{18}-y^{18}$ by x^3+y^3.

27. Shew that $(x^3+y^3)^n+\{3xy(x+y)\}^n$ is divisible by $(x+y)^3$ if n be an odd integer.

28. Shew that $(x+4)^{3n}-(7x+22)^n$ is divisible by $(x+2)\times(x+3)(x+7)$.

29. Shew that $(ab)^n-(bc)^n+(cd)^n-(da)^n$ is always divisible by $ab-bc+cd-ad$.

30. Shew that $(x^2+y^2+z^2)^{2n+1}+2.4^n(xy+yz+zx)^{2n+1}$ is divisible by $(x+y+z)^2$.

31. $(a^2+ab+b^2)^n+(a^2-ab+b^2)^n$ by $2(a^2+b^2)$ if n is odd.

32. If $a^{2n+1}+b^{2n+1}$ be exactly divisible by $a^{21}+b^{21}$, shew that $n-10$ is a multiple of 21.

33. If a^n+b^n be divisible by a^m+b^m, then $n-m$ is a multiple of $2m$.

34. If x^a+y^a be divisible by x^b+y^b, then $a-b$ is a multiple of $2b$.

35. Prove that $\frac{a^n-1}{a-1}-\frac{b^n-1}{b-1}$ is divisible by $a-b$.

36. If $a^{2n+1}+b^{2n+1}$ be divisible by a^m+b^m, prove that $n-\frac{m-1}{2}$ is a multiple of m.

When n is any positive integer,

37. Prove that 4^n-3n-1 is divisible by 9.

38. Shew that $5(5^n-1)-4n$ is divisible by 16.

39. Shew that $11^n-10n-1$ ends in two cyphers.

40. Shew that $x^n-(x-1)n-1$ is divisible by $(x-1)^2$.

41. Shew that $6^{2n}-3^{2n}$ is divisible by 3^{2n+1}.

When n is an *even* integer, shew that

42. 4^n+5n-1 is divisible by 25.

43. 7^n+8n-1 is divisible by 64.

44. $9^n+10n-1$ ends in two cyphers.

45. $x^n+(x+1)n-1$ is divisible by $(x+1)^2$.

46. If n be any positive integer, prove that $x^n-na^{n-1}x+(n-1)a^n$ is divisible by $(x-a)^2$.

47. Shew that $4.2^{4n}+2^{2(n+1)}+1$ is divisible by 9.

48. Shew that $(1-x)^{2n}-(4-7x-x^2)^n$ is divisible both by $2x-1$ and $x+3$, n being any positive integer.

49. $x^4+2^{2n}(x^2+1)^n$ is divisible by $(x^2+2)^2$ if n is an odd integer.

50. $3\{(a+b)^{2n+1}-a^{2n+1}-b^{2n+1}\}$ is divisible by $(a+b)^3-a^3-b^3$.

51. $(1+c)^n(1+b)^n(b+c)^n+b^nc^n$ is divisible by $(b+c+1)(bc+b+c)$, n being an odd integer.

52 $(a+b)^n(b+c)^n(c+a)^n+a^nb^nc^n$ is divisible by $(a+b+c)(ab+ac+bc)$, n being an odd integer.

53. $(a+b+c)^{3n}-3^n(a+b)^n(b+c)^n(c+a)^n$ is divisible by $a^3+b^3+c^3$.

54. If $a^{2n+1}+b^{2n+1}$ be divisible by $a^{m+1}+b^{m+1}$, then $\frac{2n-m}{m+1}$ is *even*.

CHAPTER VIII.

FACTORS.

63. From Chapter V it is seen, that a product is *composed* of factors, one of which is the multiplicand and the other, the multiplier. A product may, therefore, be called a compound of *elements* which are its component factors. When these factors have to be found out, we have to *resolve* or *decompose* the product. The process by which this is effected, is called the Resolution of Expressions into their component factors.

Expressions of the form $a^2 - b^2$ or those which can be put into that form.

We know that $a^2 - b^2 = (a+b)(a-b)$.

Ex. 1. $a^8 - b^8 = (a^4)^2 - (b^4)^2 = (a^4 + b^4)(a^4 - b^4)$.

Again $a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 + b^2)(a^2 - b^2)$

$= (a^2 + b^2)(a+b)(a-b)$

$\therefore$ the factors of $a^8 - b^8$ are $(a^4 + b^4)(a^2 + b^2)(a+b)(a-b)$.

Ex. 2. $a^4 + a^2b^2 + b^4 = (a^4 + 2a^2b^2 + b^4) - a^2b^2 = (a^2 + b^2)^2 - (ab)^2 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$.

Ex. 3. $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - (2x)^2$
$= (x^2 + 2 + 2x)(x^2 + 2 - 2x)$.

Note.—In the 2nd and 3rd examples, the given expression is made a perfect square by adding a perfect square to it.

Ex. 4. $a^2 - b^2 - 2bc - c^2 = a^2 - (b^2 + 2bc + c^2) = a^2 - (b+c)^2$
$= \{a + (b+c)\}\ \{a - (b+c)\}\ = (a + b + c)(a - b - c)$.

Ex. 5. $4a^2b^2 - (a^2 + b^2 - c^2)^2 = (2ab)^2 - (a^2 + b^2 - c^2)^2$

$= \{2ab + (a^2 + b^2 - c^2)\}\ \{2ab - (a^2 + b^2 - c^2)\}$

$= \{(a^2 + b^2 + 2ab) - c^2\}\ \{c^2 - (a^2 + b^2 - 2ab)\}$

$= \{(a+b)^2 - c^2\}\ \{c^2 - (a-b)^2\}$

$= \{(a+b) + c\}\ \{(a+b) - c\}\ \{c + (a-b)\}\ \{c - (a-b)\}$

$= (a + b + c)(a + b - c)(c + a - b)(c - a + b)$.

Ex. 6. $x^4 - 18x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 - 16x^2y^2$
$= (x^2 - y^2)^2 - (4xy)^2 = (x^2 - y^2 + 4xy)(x^2 - y^2 - 4xy)$.

EXERCISE 20.—(See *Exercise* 11).

Resolve into factors :—

1. x^4+x^2+1. 2. x^8+x^4+1. 3. $x^4+x^2y^2+y^4$.
4. a^8-81. 5. $x^{16}-256$. 6. $x^8+x^4y^4+y^8$.
7. x^4+64. 8. $4a^4+81$. 9. a^4+4b^4.
10. x^4+2x^2+9. 11. x^4-7x^2+9. 12. $4a^4+3a^2+9$.
13. $9x^4-33x^2+16$. 14. x^4+8x^2+144.
15. $9x^4-x^2+16$. 16. $9a^4-19a^2x^2+25x^4$.
17. $9x^4-25x^2+16$. 18. $4b^4+625$.
19. $x^2+4y^2-9z^2-4xy$.
20. $16a^2b^2-(a^2+4b^2-c^2)^2$.
21. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.
22. $a^2-2ab-c^2+2bc$.
23. $a^2-b^2-c^2+d^2-2(ad-bc)$.
24. $a^4-b^4+c^4-2a^2c^2+4ab^2c$.

64. Expressions of the form a^3+b^3 or a^3-b^3.

We know that $a^3\pm b^3=(a\pm b)(a^2\mp ab+b^2)$.

Ex. 1. $a^6-b^6=(a^3)^2-(b^3)^2=(a^3+b^3)(a^3-b^3)$
$=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

Ex. 2. $8+x^3=2^3+x^3=(2+x)(2^2-2x+x^2)$
$=(2+x)(4-2x+x^2)$.

Ex. 3. $27-x^3=3^3-x^3=(3-x)(3^2+3x+x^2)$
$=(3-x)(9+3x+x^2)$.

Ex. 4. $a^3+3a^2b+3ab^2+2b^3=a^3+b^3+3ab(a+b)$
$+b^3=(a+b)^3+b^3$
$=(a+b+b)\{(a+b)^2-b(a+b)+b^2\}$
$=(a+2b)(a^2+ab+b^2)$.

EXERCISE 21.—(See *Exercise* 13).

Resolve into factors :—

1. x^3-8. 2. a^3+27. 3. $a^3x^2y+27x^2y^4$.
4. $x^4-2xy^3+y^4$. 5. $x^4-2x^3y+3xy^3$.
6. $(a^2-bc)^3+8b^3c^3$. 7. x^6+x^3-2.
8. a^3-3a^2+3a+7. 9. $(a+b)^3-(b+c)^3$.
10. $81a^3-375$. 11. $a^3+3a^2+3a-26$.
12. $2a^3+3a^2+3a+1$. 13. $(x+y+z)^3-x^3$.

65. Expressions of the form x^2+px+q.

We know that $(x+a)(x+b)=x^2+(a+b)x+ab$.

$$(x-a)(x-b)=x^2-(a+b)x+ab.$$
$$(x+a)(x-b)=x^2+(a-b)x-ab.$$
$$(x-a)(x+b)=x^2-(a-b)x-ab.$$

To resolve an expression of the form x^2+px+q into factors, we have to find by trial two quantities a and b such that $a+b=p$ and $ab=q$.

Ex. 1. $x^2+7x+10$. Here $a+b=7$ and $ab=10$, therefore $a=5$ and $b=2$.

$\therefore\ x^2+7x+10=x^2+(5+2)x+5\times2=(x+5)(x+2)$.

Ex. 2. $x^2-12x+35$. Here $a+b=-12$ and $ab=35$, therefore $a=-7$ and $b=-5$.

$\therefore\ x^2-12x+35=x^2-(7+5)x+7\times5=(x-7)(x-5)$.

Ex. 3. $x^2+3x-40$. Here $a+b=3$ and $ab=-40$, therefore $a=8$ and $b=-5$.

$\therefore\ x^2+3x-40=x^2+(8-5)x-8\times5=(x+8)(x-5)$.

Ex. 4. $x^2-5x-36$. Here $a+b=-5$ and $ab=-36$, therefore $a=-9$ and $b=4$.

$\therefore\ x^2-5x-36=x^2-(9-4)x-9\times4=(x-9)(x+4)$.

Ex. 5 $x^2+(c-a)x+(a-b)(b-c)$.

Here the sum of $-(a-b)$ and $-(b-c)=c-a$ and their product $=(a-b)(b-c)$.

$\therefore\ x^2+(c-a)x+(a-b)(b-c)=x^2-(a-b+b-c)x+(a-b)\times(b-c)=\{x-(a-b)\}\ \{x-(b-c)\}=(x-a+b)(x-b+c)$.

Ex. 6. $(x^2-x)^2-(x^2-x)-6$.

Putting y for x^2-x, we have $y^2-y-6=y^2-(3-2)y-3\times2=(y-3)(y+2)=(x^2-x-3)(x^2-x+2)$.

66. Expressions of the form ax^2+bx+c.—When a is a perfect square, the factors may be found by the method of the previous Article, or by writing the given expression as the difference of two squares.

When a is not a perfect square, if we multiply and divide ax^2+bx+c by a, we get $\frac{1}{a}\{a\times ax^2+abx+ca\}=\frac{1}{a}(a^2x^2+abx+ac)=\frac{1}{a}\{(ax)^2+b(ax)+ac\}=\frac{1}{a}(y^2+by+ac)$, where $y=ax$.

Now the expression $y^2+by+ac$ may be resolved by the method of the previous Article.

Ex. 1. $4x^2+14x+12=(2x)^2+7(2x)+12$
$=(2x)^2+(4+3)(2x)+4\times 3=(2x+4)(2x+3)$,

or thus $4x^2+14x+12=4x^2+14x+\frac{49}{4}+12-\frac{49}{4}=\left(2x+\frac{7}{2}\right)^2$
$-\left(\frac{1}{2}\right)^2=\left(2x+\frac{7}{2}+\frac{1}{2}\right)\left(2x+\frac{7}{2}-\frac{1}{2}\right)=(2x+4)(2x+3).$

Ex. 2. $9x^2+30x+24=9x^2+30x+25-1$
$=(3x+5)^2-1^2=(3x+5+1)(3x+5-1)$
$=(3x+6)(3x+4)=3(x+2)(3x+4).$

Ex. 3. $2x^2+5x+2$. Multiplying and dividing by 2, we have $2x^2+5x+2=\frac{1}{2}\{2\times 2x^2+2\times 5x+2\times 2\}$
$=\frac{1}{2}\{(2x)^2+5(2x)+4\}=\frac{1}{2}(y^2+5y+4)$ where $y=2x$
$=\frac{1}{2}(y+4)(y+1)=\frac{1}{2}(2x+4)(2x+1)=(x+2)(2x+1).$

Ex. 4. $8x^2-26x+15$. Multiplying and dividing by 8, we have $8x^2-26x+15=\frac{1}{8}\{8\times 8x^2-8\times 26x+8\times 15\}$
$=\frac{1}{8}\{(8x)^2-26\times 8x+120\}=\frac{1}{8}(y^2-26y+120)$, where $y=8x$
$=\frac{1}{8}(y-20)(y-6)=\frac{1}{8}(8x-20)(8x-6)=(2x-5)(4x-3).$

Note.—See the Chapter on Quadratic Equations for the General method of resolving quadratic expressions.

EXERCISE 22.

Resolve into factors :—

1. x^2+x-6. 2. $x^2+25x+144$. 3. x^2-6x+5.

4. $x^2-8xy+12y^2$. 5. $49x^2+49xy+6y^2$.

6. $1+15x+14x^2$. 7. $x^2+(a+2b)x+2ab$.

8. $x^2-(2a+b)x+(a^2+ab)$. 9. $x^2-2bx+b^2-a^2$.

10. $x^2-2(a^2+b^2)x+(a^2-b^2)^2$.

11. $x^2-(a+b+c)x+ab+ac$.

12. $x^2+\left(\frac{a}{b}+\frac{b}{a}\right)x+1.$ 13. $x^2+2ax+a^2-(b-c)^2.$

14. $9(x+1)^2-3(x+1)-6.$

15. $x^2+c(a+b)x-ab(a-c)(b+c).$

16. $x^2+(a+b-c)x-ac-bc.$

17. $(x^2+5x)^2-8(x^2+5x)-84.$ 18. $20x^2+221x+600.$

19. $2x^2+x-1.$ 20. $6x^2-13x+6.$

21. $2x^2+5xy-12y^2.$ 22. $6x^2-19a^2x+10a^4$

23. $24a^2x^2-14ax^3-3x^4.$

24. $(a^2-3a)^2-3(a^2-3a)-4.$

25. $(x^2+2x)^2-(x^2+2x)-2.$ 26. $m^2+mn-30n^2,$

27. $a^2+ab-12b^2.$ 28. $x^4+3x^2-28.$

29. $a^6-10a^3+16.$ 30. $a^8-11a^4-80.$

31. $a^6+7a^3-8.$ 32. $2x^2+x-15.$

33. $6a^2-a-15.$ 34. $8x^2-6x-9.$

35. $10a^2-41ab+21b^2.$ 36. $12m^2-mn-20n^2.$

37. $x^2+(a+b)^2x+ab(a+b)^2.$

38. $x^2+(a-b)^2x-ab(a-b)^2.$

39. $x^2+(a+b)x+(a-2b)(b-2a).$

40. $(x-y)^2+3(y-z)(x-y)-(2y-z)(2z-y).$

67. Expressions which contain a Single Power only of some Letter or Quantity.—Such expressions can be arranged in two groups, one group containing all the terms in which this letter occurs, and the other group containing the remaining terms.

Ex. 1. $ab+cd+ac+bd$. Arranging the expression in two groups, one containing the terms in which b occurs and the other containing the remaining terms, we have $ab+cd+ac+bd$ $=(ab+bd)+(cd+ac)=b(a+d)+c(d+a)=(a+d)(b+c).$

Ex. 2. $a^2bx-a^2c-b^2cx+bc^2$. Here x occurs in the first power only; hence we have the expression $=(a^2bx-b^2cx)$ $-(a^2c-bc^2)=bx(a^2-bc)-c(a^2-bc)=(a^2-bc)(bx-c).$

Ex. 3. $ax+bx-ay-by.$

The expression $=x(a+b)-y(a+b)=(a+b)(x-y).$

EXERCISE 23.

Resolve into factors :—

1. $ac-ad+bc-bd.$ 2. $abx^2+(b-a)xy-y^2.$
3. $2x^2+4ax+6bx+12ab.$ 4. $2a^3-a^2b+4abx-2b^2x.$
5. $(2x^2-3a^2)y+(2a^2-3y^2)x.$
6. $ax^3+(a+b)x^2+(a+b)x+a.$
7. $1-x-y+xy$ 8. $ax+a+x+1.$
9. $a^2-ab-ac+bc.$ 10. $p^2+pq+pr+rq.$
11. $4x-by-4y+bx.$ 12. $a^2+a+b+ab.$

68. Expressions which, when arranged in groups of two or more terms, have a factor common to all the groups.

Ex. 1. $x^3-x^2y+xy^2-y^3 = (x^3-y^3)-xy(x-y) = (x-y)\{x^2+xy+y^2-xy\} = (x-y)(x^2+y^2),$
or thus : the expression $=(x^3-x^2y)+(xy^2-y^3)=x^2(x-y)+y^2(x-y)=(x-y)(x^2+y^2).$

Ex. 2. $ab(x^2+y^2)+xy(a^2+b^2)=abx^2+aby^2+xya^2+xyb^2$ $=(abx^2+xya^2)+(aby^2+xyb^2)=ax(bx+ay)+by(ay+bx)$ $=(bx+ay)(ax+by).$

EXERCISE 24.

Resolve into factors :—

1. $(3x^2-4b^2)a+(3a^2-4x^2)b.$ 2. $x^4+x^2y^2-y^2z^2-z^4$
3. $x^2-y^2-z^2+2yz+x+y-z.$ 4. $x^2-2xy+y^2-x+y.$
5. $a(a+c)-b(b+c).$ 6. $x^4+2x^3a-2xa^3-a^4$
7. $x^4-2x^3y+2xy^3-y^4.$ 8. $bx-ac-cx+ab.$
9. $a^2-b^2-c^2-2bc+a-b-c.$ 10. $a^3+a^2b+ab^2+b^3.$
11. $a^3-2a^2b+ab^2-2b^3.$ 12. $xy(1+z^2)+z(x^2+y^2).$
13. $a(b^2+c^2-a^2)+b(c^2+a^2-b^2).$
14. $x(xy+x+1)-y(xy+y+1).$
15. $a(a+1)+2ab+b(b+1).$
16. $b(a^2+c^2-b^2)+c(a^2+b^2-c^2)$

69. Expressions of the form $a^3+b^3+c^3-3abc$.

We know that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc).$

Ex. 1. $a^3-b^3+c^3+3abc=a^3+(-b)^3+c^3-3a(-b)(c)$
$=(a-b+c)\{a^2+b^2+c^2-a(-b)-ac-c(-b)\}$
$=(a-b+c)(a^2+b^2+c^2+ab-ac+bc).$

Ex. 2. $m^3-n^3+1+3mn=m^3+(-n)^3+1^3-3m(-n)(1)$
$=\{m+(-n)+1\}\ \{m^2+(-n)^2+1^2-m(-n)-m(1)-1(-n)\}$
$=(m-n+1)(m^2+n^2+1+mn-m+n).$

EXERCISE 25.

Resolve into factors :—

1. $b^3+c^3-a^3+3abc.$
2. $c^3+36c+37.$
3. $28b^3-9b^2c+c^3.$
4. $9a^3+6a^2b-b^3.$
5. $a^3-b^3-c^3-3abc.$
6. $x^6+y^6-z^6+3x^2y^2z^2.$
7. $\dfrac{a^3}{b^3}+\dfrac{b^3}{c^3}+\dfrac{c^3}{a^3}-3.$
8. $(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a).$
9. $x^3+\dfrac{1}{x^3}-2.$
10. $p^3-\dfrac{1}{27p^3}-2.$

70. If the algebraic sum of the co-efficients of an expression in x be equal to 0, then it has a factor x−1.

Ex. 1. ax^2+bx+c will have $x-1$ as a factor if $a+b+c=0.$
$ax^2+bx+c=ax^2+bx+c-(a+b+c)$
$=a(x^2-1)+b(x-1)=(x-1)(ax+a+b).$

Ex. 2. $4x^3+3x^2-7.$ Here the sum of the co-efficients $=4+3-7=0$; therefore it will have $x-1$ as a factor.

Now $4x^3+3x^2-7=4x^3-4+3x^2-3=4(x^3-1)+3(x^2-1)$
$=(x-1)\{4(x^2+x+1)+3(x+1)\}.$

71. If the sum of all the odd co-efficients of an expression in terms of x = the sum of all the even ones, then it has a factor x+1.

Ex. 1. ax^2+bx+c will have $x+1$ as a factor if $a+c=b$, *i.e.*, if $a+c-b=0.$

Now $ax^2+bx+c=ax^2+bx+c-(a+c-b)$
$=a(x^2-1)+b(x+1)=(x+1)(ax-a+b).$

Ex. 2. $3x^3+5x^2+6x+4.$ Here $3+6$ which is the sum of the odd co-efficients $=5+4$ which is the sum of the even ones ; therefore $x+1$ is a factor of the expression.

$$\text{Now } 3x^3+5x^2+6x+4 = 3x^2(x+1)+2x(x+1)+4(x+1)$$
$$=(x+1)(3x^2+2x+4).$$

Note.—(1) If any powers of x are missing, they must be considered as present with zero co-efficients.

(2) An expression in x will have x^2-1 for a factor, if the sum of the odd co-efficients and the sum of the even ones be each equal to 0.

EXERCISE 26.

Resolve into factors :—

1. $3a^3-2a-1$.
2. $2x^3-5x^2+7x-4$.
3. $2x^3+x^2-2x-1$.
4. $3x^4-7x^3+9x^2-12x+7$.
5. $x^5-5x^3+5x^2-1$.
6. x^3-3x^2+3x-1.
7. $3a^3-5a^2+3a-1$.
8. a^3+3a^2-4.
9. $3x^4-2x^3+3x^2-8$.
10. $4a^3-6a^2+5a-3$.
11. $11x^3+7x^2-5x-1$.
12. $7x^4+4x^3-2x^2-x-2$.
13. $2-a-a^3$.
14. $c^3+36c+37$.
15. x^3-x^2+x-1.
16. $ax^3+bx^2+cx-(a+b+c)$
17. x^4-x^2+7x+5.
18. $6x^3-7x^2+1$.
19. x^3+3x^2+5x+3.
20. x^3+x^2+x-3.

72. Expressions of the form $(a+b+c)^3-a^3-b^3-c^3$.

$$(a+b+c)^3-a^3-b^3-c^3=(a+b+c)^3-a^3-(b^3+c^3)$$
$$=(b+c)\{(a+b+c)^2+a(a+b+c)+a^2-(b^2-bc+c^2)\}$$
$$=(b+c)\{a^2+b^2+c^2+2ab+2bc+2ac+a^2+ab+ac+a^2$$
$$-b^2+bc-c^2\}$$
$$=(b+c)\{3a^2+3ab+3bc+3ac\}=3(b+c)\{a(a+b)$$
$$+c(a+b)\}$$
$$=3(b+c)(a+b)(a+c)=3(a+b)(b+c)(c+a).$$

Ex. 1. $(x+2y+2z)^3-(x+y)^3-(y+z)^3-z^3$.

Putting a for $x+y$, b for $y+z$ and c for z, we have the expression

$$=(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$$
$$=3(x+y+y+z)(y+z+z)(z+x+y)=3(x+2y+z)(y+2z)$$
$$\times(x+y+z).$$

Ex. 2. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.

Putting x for $b+c-a$, y for $c+a-b$ and z for $a+b-c$, we have the given expression$=(x+y+z)^3-x^3-y^3-z^3=3(x+y)\times(y+z)(z+x)$

$$=3(b+c-a+c+a-b)(c+a-b+a+b-c)(a+b-c+b$$
$$+c-a).$$
$$=3(2c)(2a)(2b)=24abc.$$

73. $(a+b)(b+c)(c+a)$

$$
\begin{aligned}
&=(a+b+c)(ab+ac+bc)-abc \quad \ldots \quad \ldots \quad (i)\\
&=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc \quad \ldots \quad \ldots \quad (ii)\\
&=a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc \quad \ldots \quad \ldots \quad (iii)\\
&=a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc \quad \ldots \quad \ldots \quad (iv)\\
&=ab(a+b)+bc(b+c)+ca(c+a)+2abc \quad \ldots \quad \ldots \quad (v)\\
&=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \quad \ldots \quad \ldots \quad (vi)\\
&=\frac{1}{3}\{(a+b+c)^3-a^3-b^3-c^3\}.
\end{aligned}
$$

See Article 51, Chapter V.

74. Symmetrical expressions :—

Ex. 1. $a^2(b-c)+b^2(c-a)+c^2(a-b)$

$$
\begin{aligned}
&=a^2(b-c)+b^2(c-b+b-a)+c^2(a-b)\\
&=a^2(b-c)-b^2(b-c)-b^2(a-b)+c^2(a-b)\\
&=(b-c)(a^2-b^2)-(a-b)(b^2-c^2)\\
&=(b-c)(a-b)\{(a+b)-(b+c)\}=(b-c)(a-b)(a-c).
\end{aligned}
$$

Note.—The letter b introduced in the binomial factor of the middle term is the one which is not found in it. We may introduce the letter a in the first term or the letter c in the last term and resolve the expression in the same way.

The expression may be arranged according to powers of a, or b, or c and resolved into factors.

Ex. 2. $a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)$

$$
\begin{aligned}
&=a^2b^2(a-b)+b^2c^2(b-a+a-c)+c^2a^2(c-a)\\
&=a^2b^2(a-b)-b^2c^2(a-b)-b^2c^2(c-a)+c^2a^2(c-a)\\
&=(a-b)(a^2b^2-b^2c^2)+(c-a)(c^2a^2-b^2c^2)\\
&=b^2(a-b)(a^2-c^2)-c^2(a-c)(a^2-b^2)\\
&=(a-b)(a-c)\{b^2(a+c)-c^2(a+b)\}\\
&=(a-b)(a-c)\{a(b^2-c^2)+bc(b-c)\}\\
&=(a-b)(a-c)(b-c)\{a(b+c)+bc\}\\
&=(a-b)(b-c)(a-c)(ab+ac+bc).
\end{aligned}
$$

Note.—See the Chapter on Miscellaneous Theorems and Examples for more information about Symmetrical expressions.

EXERCISE 27.

Resolve into factors :—

1. $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$.
2. $(3+a+b+c)^3-(1+a)^3-(1+b)^3-(1+c)^3$.
3. $(a^2+b^2+c^2-ab-ac-bc)^3-(a^2-bc)^3-(b^2-ac)^3$
$-(c^2-ab)^3$.

4. $(3-x-y-z)^3-(1-x)^3-(1-y)^3-(1-z)^3$.
5. $ab(a-b)+bc(b-c)+ca(c-a)$.
6. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.
7. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.
8. $ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)$.
9. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.
10. $a^4(b-c)+b^4(c-a)+c^4(a-b)$
11. $a^2b^2(a^2-b^2)+b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)$.
12. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$.
13. $a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)$.
14. $(a+b)^2(b-a)+(b+c)^2(c-b)+(c+a)^2(a-c)$.
15. $b^2(1-b^2)+b^2c^2(b^2-c^2)+c^2(c^2-1)$.
16. $\frac{a}{c}\left(\frac{a}{b}-\frac{b}{c}\right)+\frac{b}{a}\left(\frac{b}{c}-\frac{c}{a}\right)+\frac{c}{b}\left(\frac{c}{a}-\frac{a}{b}\right)$.
17. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.
18. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.
19. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
20. $a^4(b^2+c^2)+b^4(c^2+a^2)+c^4(a^2+b^2)+2a^2b^2c^2$.
21. $x^2(y^2+z^2)^2+y^2(z^2+x^2)^2+z^2(x^2+y^2)^2-4x^2y^2z^2$.
22. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)-1$.
23. $\left(x^3+\frac{1}{x}+\frac{1}{x^2}\right)\left(x+x^2+\frac{1}{x^3}\right)-1$.
24. $4c^2(a+b-c)+4b^2(c+a-b)+4a^2(b+c-a)$
$\quad -4(a+b-c)(b+c-a)(c+a-b)$.
25. $s^3-(s-a)^3-(s-b)^3-(s-c)^3$, if $2s=a+b+c$.

75. The following method may be applied to *the resolution of homogeneous expressions of the second degree* involving three or more quantities.

Ex. 1. $7a^2-ab-5ac-6b^2-8bc-2c^2$.

First, suppose $a=0$; there remains $-6b^2-8bc-2c^2$
$=(6b+2c)(-b-c)$.. .. I.

Secondly, suppose $b=0$; there remains $7a^2-5ac-2c^2$
$=(7a+2c)(a-c)$..... II.

Thirdly, suppose $c=0$; there remains $7a^2-ab-6b^2$
$=(7a+6b)(a-b)$..... III.

Combining the results of I, II and III, we find that $7a$, $2c$ and $6b$ go in one factor, and $a, -b$ and $-c$ go in the other. Therefore the required factors are $(7a+6b+2c)(a-b-c)$.

Ex 2. $a^2+2b^2+2c^2+3ab+3ac+5bc$.

First, let $a=0$; there remains $2b^2+2c^2+5bc$
$=(b+2c)(c+2b)$. . . . . . I.

Secondly, let $b=0$; there remains a^2+2c^2+3ac
$=(a+2c)(a+c)$. . . . . II.

Thirdly, let $c=0$; there remains a^2+2b^2+3ab
$=(a+2b)(a+b)$. . . III.

Combining the results of I, II and III, we find that a, b and $2c$ go in one factor, and a, $2b$ and c go in the other. Therefore the required factors are $(a+b+2c)(a+2b+c)$.

Note.—If the expression involves four letters, suppose *two of them at a time to be zero*; if it involves five letters, suppose *three of them at a time to be zero*; and so on

EXERCISE 28.

Resolve into factors :—

1. $6x^2+21xy+18y^2+28z^2+45yz+26xz$.
2. $6a^2+13ab-5ac-5b^2+13bc-6c^2$.
3. $2x^2+7xy+7xz+6y^2+11yz+3z^2$.
4. $10a^2-8ab-11ac-24b^2-58bc-35c^2$.
5. $a^2-2b^2+c^2+ab+2ac+bc$.
6. $a^2+b^2+c^2+2ab+2ac+2bc$.
7. $2x^2+2y^2-z^2+5xy+xz-yz$.
8. $2a^2+2b^2+c^2+5ab+3ac+3bc$.
9. $2a^2-b^2+2c^2+ab+5ac-bc$.
10. $3x^2+4z^2-6xy-8xz+8yz+3y^2$.

Resolve into factors, applying the same principle :—

11. $2x^2-9ax+10x-18a^2+45a-28$.
12. $3x^2+7xy-4x+4y^2-7y-15$.
13. $6a^2+11ab-13a-10b^2+34b-28$.
14. $3x^2-3xy+25x-6y^2+40y-50$.
15. $2a^2+2b^2-9+5ab-3a+3b$.
16. $14x^2+33xy+46x+18y^2+42y+12$.

Resolve into factors :—

17. $3x^2-2y^2-6z^2-5xy-3xz-8yz$.

18. $xy+2x^2-3y^2+4yz+xz-z^2$.

19. $2x^2-9xz-5xy+4z^2-8yz-12y^2$.

20. $a^2+b^2+c^2+d^2+2ab+2ac-2ad+2bc-2bd-2cd$.

76. A great many apparently difficult expressions can be readily factorized, *when arranged according to powers of some contained letter or quantity.* This is an extension of the method employed in Articles 67 and 68.

Ex. $a^3(b-c)+b^3(c-a)+c^3(a-b)$

$=a^3b-a^3c+b^3c-b^3a+c^3a-c^3b$

$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$ (arranged according to descending powers of a)

$b-c$ is common

$=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$

$=(b-c)\{b^2(c-a)+bc(c-a)-a(c^2-a^2)\}$ (arranged according to powers of b)

$c-a$ is common

$=(b-c)(c-a)\{b^2+bc-a(c+a)\}$

$=(b-c)(c-a)\{c(b-a)+b^2-a^2\}$ (arranged according to powers of c)

$=(b-c)(c-a)(b-a)(c+b+a)$.

77. To prove that $a^3+b^3+c^3=3abc$, when $a+b+c=0$.

Since $a+b+c=0$, $\quad a+b=-c$

$\therefore (a+b)^3=-c^3$

or $a^3+b^3+3ab(a+b)=-c^3$

$\therefore a^3+b^3+3ab(-c)=-c^3$, since $a+b=-c$

$\therefore a^3+b^3-3abc=-c^3 \quad \therefore \quad a^3+b^3+c^3=3abc$.

Hence, when the sum of three quantities is zero, the sum of their cubes is equal to three times their product.

Ex. Resolve into factors $(a-b)^3+(b-c)^3+(c-a)^3$.

Since $a-b+b-c+c-a=0$, $(a-b)^3+(b-c)^3+(c-a)^3$
$=3(a-b)(b-c)(c-a)$.

Putting x for $a-b$, y for $b-c$ and z for $c-a$, we have the expression $=x^3+y^3+z^3$; and $x+y+z=a-b+b-c+c-a=0$
$\therefore x^3+y^3+z^3=3xyz$,

i.e., $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

EXERCISE 29.

Find the factors of :—

1. $(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3$.
2. $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$.
3. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.
4. $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$.
5. $(x+a)^3-(x-b)^3-(a+b)^3$.
6. $(s-a)^3+(s-b)^3-c^3$, where $2s=a+b+c$.
7. $a^3\left(\frac{b}{c}-\frac{c}{b}\right)^3+b^3\left(\frac{c}{a}-\frac{a}{c}\right)^3+c^3\left(\frac{a}{b}-\frac{b}{a}\right)^3$.
8. $a^9(b-c)^3+b^9(c-a)^3+c^9(a-b)^3$, if $a+b+c=0$.
9. $(2a-b)^3+(2b-c)^3+(2c-a)^3$, if $a+b+c=0$.
10. $(x-2y)^3+(2y-1)^3+(1-x)^3$.
11. $(ax-by)^3+(by-cz)^3+(cz-ax)^3$.
12. $(3a-s)^3+(3b-s)^3+(3c-s)^3$, if $s=a+b+c$.
13. $a^3(bz-cy)^3+b^3(cx-az)^3+c^3(ay-bx)^3$.

CHAPTER IX.

EXAMINATION PAPERS.

FIRST SERIES ON CHAPTERS I—VIII.

I.

1. State the rule for the removal of brackets from algebraical expressions, and simplify $2[4x - \{2y + (2x - y) - ($

2. Sum up $3a^2 - 4ab + 7a^2b^2 + 2ab^3$, $2b(a-b) - a^2(b^2 + 4ab)$, $ab - 2a^2 + b^2(2 + a^2)$ and $-b(6a^2b - b - 2a - 2a^3)$.

3. Multiply (1) $2x^3 - 3x^2 + 4x - 5$ by $7x + 9$ and (2) $\frac{2}{3}a^3 - \frac{4}{5}a^2x - \frac{1}{2}x^3$ by $\frac{3}{5}a - 2x$.

4. Find the continued product of $(x-a)(x-b)(x-c)$ and hence deduce $(\frac{1}{2}m - \frac{1}{3}n)^3$.

5. If $a = ·05$, $b = ·06$ and $c = -·11$, find (1) $a^2 - b^2 - 2bc - c^2$; (2) $a^3 + b^3 + c^3 - 3abc$.

6. Divide $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$ by $x - c$ and the quotient by $x - b$.

7. Find the co-efficient of x^4 in the product of $(x^4 - ax^3 + bx^2 - cx + d)$ and $(x^2 + px + q)$.

8. Resolve into factors (1) $x^2 - 3x - 70$; (2) $x^3 + x + 2$.

9. Write down the terms in the quotient of $\dfrac{x^8 - a^8}{x - a}$.

10. Divide 1 by $1 - x$ to six terms.

II.

1. Define *a factor*, *a term* and *a co-efficient*; and find the co-efficient of x^6 in $(1 + x + x^2 + x^3)^2$.

2. Find the value of $(2b + c)(b - c) + (2c + a)(c - a) + (2a + b) \times (a - b)$ when $a = 1$, $b = 2$ and $c = 3$.

3. Remove the brackets and find the value of—

$a - [2a - 3b - \{4a - 5b - 6c - (7a - \overline{8b + 9c + 10d})\}]$ when $a = 1$, $b = \frac{1}{2}$, $c = 3$ and $d = \frac{1}{5}$.

4. Find the continued product of (1) $x + \sqrt{x} + 1$, $x^2 - x - 1$ and $x - \sqrt{x} + 1$; (2) $a^2 + 2a + 2$, $a^4 - 4$ and $a^2 - 2a + 2$.

5. Divide the difference between $(x+p)(x+q)(x+r)$ and $(y+p)(y+q)(y+r)$ by $x - y$.

6. When is $x^p + y^p$ divisible by $x^q + y^q$? Write down the first four terms in the quotient.

7. Resolve into factors :—

(1) a^8+a^4+1;

(2) $x^2(y-z)+y^2(z-x)+z^2(x-y)$,

(3) $x^4+(a^2-c^2)x^2+(b^2-c^2)(a^2-b^2)$.

8. Collect the co-efficients of the different powers of x in $(px^3+qx^2+rx+s)(sx^3+rx^2+qx+p)$ and shew that the sum is a perfect square.

9. Shew that $(a+2b)^{2n}-(b+2a)^{2n}$ is divisible by $(a+b)$ and by $(a-b)$.

10. From the relation $a^m \div a^n = a^{m-n}$, deduce $a^0 = 1$.

III.

1. What is a *homogeneous* expression? Shew that the product of two homogeneous expressions will also be a homogeneous expression.

2. Add together the squares of $a+b-2c$, $b+c-2a$ and $c+a-2b$, and subtract the sum from the square of $a+b+c$.

3. Multiply (i) $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{c}-\frac{b}{a}-\frac{c}{b}$ by $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$;

(ii) $(a+b-c)(b+c-a)(c+a-b)$.

4. Divide (i) $a^3-b^3+c^3+3abc$ by $a-b+c$,

(ii) $5(3a^4+2)-9a(1-3a)-13a^3+6a^2$ by $5a^2+2-a$.

5. If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$, find the value of—

$$\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd+ac}-\sqrt{\left(\frac{b^3}{a^3}-\frac{a^3}{c^3}\right)}.$$

6. Show that $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$ $=4a^2b^2$ if $a^2+b^2=c^2$.

7. If $a^{2n+m}+b^{2n+m}$ be divisible by $a^{m+10}+b^{m+10}$, prove that $\frac{n-5}{m+10}$ is an integer.

8. Shew that $5.25^n\{11.121^n+9.81^n\}$ is divisible by 100 if n is a positive integer.

9. Resolve into factors: (1) $a(b^3-c^3)+bc(c^2-b^2)+a^3(c-b)$

(2) $(a-b+c)^3-a^3-c^3+b^3$.

10. State the remainder when $(x-a)(x-b)(x-c)$ is divided by $x-p$.

IV.

1. State the rule for subtracting one algebraical expression from another.

Prove that $a-(b-c)=a-b+c$.

2. If $a=3$, $c=-4$, find the value of—

$$21\left(\frac{2a+4c}{3}-\frac{4a-c}{7}\right)-12\left[\frac{2a-c}{6}-\left\{\frac{5a-7c}{4}-\frac{1}{3}\left(\frac{a}{2}-\frac{5a-c}{4}\right)\right\}\right].$$

3. The product of two expressions is $10x^4+14x^3-11x^2+15x-3$ and one of them is $2x^2+4x-1$; find the other.

4. Prove that $(x-y)^3+(y-z)^3+(z-x)^3$ is exactly divisible by $(x-y)$.

5. Divide without removing the brackets $(a+b)(b+c)x^5+\{(b+c)^2+(a+b)^2\}x^4+(b+c)(a-b)x^2+(a+b)(a+c)x-(a+c)\times(b+c)$ by $(b+c)x^2+(a+b)x-(b+c)$.

6. Find the co-efficient of a^8 in the square of—

$$1-2a+3a^2+4a^4-a^7.$$

7. If $x=a-b$, $y=b-c$, $z=c-a$, find the value of $x^3+y^3+z^3-3xyz$ and of $x^2(y+z)+y^2(x+z)+z^2(x+y)+3xyz$.

8. Divide $1-x$ by $1+x$ to 5 terms.

9. Resolve into factors :—

(i) $a^3-(b-c)^3$; (ii) $10x^2-7x-3$;

(iii) $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.

10. Shew that $(x+a+b)^3+(x+a-b)^3$ is divisible by $x+a$.

V.

1. State the "Rule of Signs" and from it deduce $(-a)^7=-a^7$ and $(-a)^{14}=+a^{14}$. Shew that $a^{2n+1}+b^{2n+1}=0$, if $a+b=0$.

2. If $a=1$, $b=2$ and $c=3$, shew that

$$\frac{a^b+c^a}{b^a+a^c}+\frac{a^a+b^b}{c^c+a^{bc}}-\frac{1}{a^c b^b c^a}=\frac{20}{a^b+b^b+c^b}.$$

3. Divide $x^3+8y^3-27z^3+18xyz$ by $x+2y-3z$.

4. The product of two expressions is $(x+2y)^3+(3x+z)^3$ and one of them is $4x+2y+z$; find the other.

5. If $a=\cdot02$, $b=\cdot08$, $c=\cdot1$, find the value of $a^3+b^3+3abc-c^3$.

6. Resolve into factors:—

(i) $x^2a-(a-b)^3-x^2b$;

(ii) $x^4+10x^3+35x^2+50x+24$;

(iii) $(x+y+z)(xy+yz+zx)-xyz$.

7. Shew that $a^n-(-1)^n b^n$ is divisible by $a+b$ and thence deduce that $49^n+16n-1$ is divisible by 8^2.

8. Find the continued product of $x-a+\sqrt{a^2-b^2}$, $x+a-\sqrt{a^2+b^2}$, $x-a-\sqrt{a^2-b^2}$ and $x+a+\sqrt{x^2+b^2}$.

9. If $a^2+b^2=1$, shew that $a^6+b^6-\frac{3}{2}(a^4+b^4)+\frac{1}{2}=0$.

10. If $x^4+ax^3+bx^2+ax+b-1$ be exactly divisible by $x-1$, shew that $a+b=0$.

VI.

1. Prove that (i) $a^m\times a^n=a^{m+n}$; (ii) $a^m\div a^n=a^{m-n}$ when m and n are positive integers and $m>n$.

2. Simplify $21\left[\frac{1}{3}(2x+4y)-\frac{1}{7}(4x-y)\right]-24\left[\frac{1}{12}(2x-y)\right.$
$$\left.-\left\{\frac{1}{8}(5x-7y)-\frac{1}{6}\left(\frac{x}{2}-\frac{5x-y}{4}\right)\right\}\right].$$

3. Multiply $(b+c)x+(c+a)y+(a+b)z$ by $(b-c)x+(c-a)y+(a-b)z$, and write the product as the difference of two squares.

4. Express in words the formulæ:—

(i) $a^2-b^2=(a+b)(a-b)$;

(ii) $a^3+b^3+3ab(a+b)=(a+b)^3$

5 Find the co-efficient of x^{10} in the quotient when 1 is divided by $1+x+x^2$.

6. If $b+c-3a=2x$, $c+a-3b=2y$ and $a+b-3c=2z$, express $(x+2a)(y+2b)(z+2c)$ in terms of a, b and c.

7. If $x^2+2ax-3b^2$ is divisible by $x-a$, without remainder shew that $a^2-b^2=0$.

8. If $a^{2n+1}+b^{2n+1}$ be divisible by a^7+b^7, shew that $n-3$ is a multiple of 7, n being a positive integer.

9. Resolve into factors: (i) $ax^2+(a-b)x-b$;
(ii) $4abx^2-2(a^2+b^2)xy+aby^2$; (iii) $abx^2-(a^2-b^2)x-ab$.

10. Shew that $(x^2+y^2)(a^2+b^2)=(ax+by)^2+(bx-ay)^2$, or $=(ax-by)^2+(bx+ay)^2$.

VII.

1. Explain the following symbols: $\pm$, $\backsim$, $>$, $<$, $\not>$, $\not<$ and $\equiv$.

2. State the rules for the insertion of brackets in an algebraical expression.

Enclose $3a-2b-c-2d+e-4f-3g-5h$ in brackets taking the terms three together.

3. Divide $y^4+ay^3+(a^2+4)y^2+(4a+1)y+5a^2$ by y^2+4 and find the value of y for exact divisibility.

4. Find the co-efficient of x^3 in $(x-1)(x^2-x-1)(x^2-2x-2)$ and in $(a-x)(b-x)(c-x)(d-x)$.

5. Multiply $a^5-3a^4x+9a^3x^2-27a^2x^3+81ax^4-243x^5$ by $a+2x$ and divide the product by $a-3x$.

6. Shew that a^n+b^n is divisible by $a+b$ when n is an *odd* integer and a^n-b^n is divisible by $a+b$ when n is an *even* integer.

Shew that $5^{2n}-1$ is divisible by 24, n being a positive integer.

7. If $x=\frac{1}{4}$, $y=\frac{1}{2}$ and $z=\frac{3}{4}$, find the value of $2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4$.

8. Resolve into factors: (i) $4+\dfrac{1}{a^4}$; (ii) $8a^3+b^3-c^3+6abc$;

(iii) $4b^2c^2-(b^2+c^2-a^2)^2$.

9. Shew that $a^3=\left(a^2+\frac{1}{4}a\right)^2-\left(a^2-\frac{1}{4}a\right)^2$.

10. Shew that $\left(x+\dfrac{1}{x}\right)\left(y+\dfrac{1}{y}\right)+\left(x-\dfrac{1}{x}\right)\left(y-\dfrac{1}{y}\right)$ $=2\left(xy+\dfrac{1}{xy}\right)$; thence find the value of $xy+\dfrac{1}{xy}$, if $x+\dfrac{1}{x}=2m$, $y+\dfrac{1}{y}=2n$, $x-\dfrac{1}{x}=2a$ and $y-\dfrac{1}{y}=2b$.

VIII.

1. Find the continued product of $(x+a)(x+b)(x+c)(x+d)$ and state the laws that hold in the products of similar binomials.

2. Find the difference between $(x-a+b)(x-b+c)(x-c+a)$ and $(x+a-b)(x+b-c)(x+c-a)$.

3. Find the value of m for which $x^4+5x^3+7x^2+mx-2m$ is exactly divisible by x^2+3x+2.

4. Simplify $[-\{-1-(-1-\overline{1-1})-\overline{1-1}\}-\overline{1-1}]$.

5. Shew that $(x^2+yz)^n-x^n(y+z)^n$ is divisible by $(x-y)\times(x-z)$.

6. Shew that $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$
$$=(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2.$$

7. Divide $(a+b)^3-(b+c)^3+(c-a)^3$ by $(a+b)(b+c)$.

8. Find the co-efficient of x^4 in $(1+2x+3x^2+4x^3)^2$.

9. Resolve into factors:—

(i) $x^9+x^6+x^3+1$; (ii) $4a^2+2b^2-6c^2+6ab-2ac+bc$;

(iii) $(x^3+1)(y^2+y+1)(y+1)-(y^3+1)(x^2+x+1)(x+1)$;

(iv) x^4+2x^2+9;

(v) $(a+b+c+d+e+f)^3-(a+d)^3-(b+e)^3-(c+f)^3$.

10. Find, *without division*, the remainder when $x^3+x^2(a+b)+2x(a+b)^2+(a+b)^3$ is divided by $x-a-b$.

IX.

1. Define *positive* and *negative quantities*; *odd and even numbers*; and *like and unlike terms*.

2. Prove that $a-b-(c-d)=a-b-c+d$.

3. If $a=\sqrt{2}$, $b=\sqrt{3}$, $c=4$ and $d=0$, find the value of $\sqrt{\{(a^2+b^2+c^2)(b^2+c^2)(b^2+d^2)\}}$.

4. Find the continued product of $(x+m)(x+n)(x+p)\times(x+q)$, and thence deduce the expansion of $(a+b)^4$.

5. Divide, *without removing brackets*, $(a+b)x^4+(a^2-1+b^2+2ab)x^3-(a+b)(1-2ab)x^2+(a-2ab+b)x-1$ by $(a+b)x-1$.

6. Simplify $(x+y+z)^3+(x+y-z)^3+(y+z-x)^3+(z+x-y)^3$ and shew that $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2=(ay-bx)^2+(cx-az)^2+(cy-bz)^2$.

7. Resolve: (i) $a^3+b^3+3ab-1$; (ii) $a^2b^2-a^2-b^2+1$;

(iii) $3(a+b)^2-2(a^2-b^2)-a(a+b)$;

(iv) $a^2-b^2+6bc-9c^2$; (v) x^3+x^2-x-1;

(vi) $x^2+2xy-a^2+2ay$; (vii) $14x^2-37x+5$.

8. Shew that $(a+b)^3-(b+c)^3+(c-a)^3=3(a+b)(b+c)\times(a-c)$.

9. If $a=\frac{25}{16}$, $b=1$ and $c=\frac{3}{4}$, prove that

$$(a-\sqrt{b})(\sqrt{a}+b)\sqrt{a-b}=\frac{3c^4}{\sqrt{a-c^2}}.$$

10. Shew that $(x+y)(y+z)(z+x)-(x+y+z)(xy+yz+zx)=-xyz$; and thence simplify $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{a}+\frac{a}{b}\right)$

$$-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right).$$

X.

1. Define a *power* of a quantity and the *index* of the power. What does a^n stand for?

Prove that $(a^m)^n=a^{mn}$ when m and n are positive integers.

2. Substitude $a+2$ for x in $x^3+6x^2+12x+8$, and simplify the result.

3. Find the value of:—

$$\frac{2a^3-b^3+c^3+a^2(b-c)+b^2(2a-c)+c^2(2a+b)}{2a^3-b^3+c^3+a^2(b-c)-b^2(2a-c)+c^2(2a+b)}\text{ when}$$

$a=1$, $b=3$ and $c=5$.

4. Multiply $3a^2+ab-b^2$ by $a^2-2ab-3b^2$ and divide the product by $a+b$.

5. Write down the terms in the quotient of—

$(x^8-y^8)\div(x^2-y^2)$; $(x^{14}+y^{14})\div(x^2+y^2)$ and $(x^6+y^6)\div(x^2+y^2)$.

6. Shew that the following expressions are divisible by $x+1$ and $x-1$:—(i) $4x^5+3x^4-2x^3-2x^2-2x-1$;

(ii) $5x^5-4x^4-2x^3+2x^2-3x+2$.

7. Resolve into factors:—(i) $16x^4-57x^2+9$;

(ii) $8m^3-10-\dfrac{27}{8m^3}$; (iii) $x(x+1)(x+2)(x+3)+1$.

8. Shew that $(a+b+c)^2+(a+b-c)^2-(a+c-b)^2$

$$-(b+c-a)^2=8ab.$$

9. If $a^{2n+1}+b^{2n+1}$ be divisible by a^5+b^5, shew that $n-2$ is a multiple of 5.

10 If ax^3+bx^2+c be divisible by $x+c$, then $ac^2=bc+1$.

CHAPTER X.

IDENTITIES.

78. When two quantities are equal to each other **for all** values of the letters involved in them, they form an *identity;* the sign $\equiv$ is placed between them.

For example, $a^2-b^2\equiv(a+b)(a-b)$, $(x+y)^3\equiv x^3+y^3+3xy(x+y)$, and $a^3+b^3+c^3-3abc\equiv(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$ are identities.

But $a^3+b^3+c^3=3abc$ when $a+b+c=0$ is a *conditional identity;* for it is true only when the condition $a+b+c=0$ is given.

To prove that an identity is true, take the left side expression and show by successive transformations that it is equal to the other.

The following are the *axioms* on which the operations with identities are based.

1. If equals be *added to* or *subtracted from equals*, the *sums* or the *remainders* are equal. Thus, if $a=b$, $c=d$, then $a+c=b+d$ and $a-c=b-d$.

2. If equal quantities be *multiplied* or *divided* by the same or equal quantities, the *products* or the *quotients* are equal.

Thus, if $a=b, c=d$, then $ac=bd$ and $\frac{a}{c}=\frac{b}{d}$.

3. Any quantity may be transposed from one side of the identity to the other by changing its sign. This follows from (1); thus if $a+b=c$, then $a=c-b$ for $(a+b-b=c-b)$.

4. If two quantities are equal, their roots or their powers are also equal; thus, if $a=b$, then $\sqrt{a}=\sqrt{b}$ and $a^2=b^2$.

79. Examples worked out.

Ex. 1. If $a+b+c=0$, shew that—

(i) $(ab+ac+bc)^2=a^2b^2+b^2c^2+c^2a^2$.

(ii) $a^2-bc=b^2-ac=c^2-ab$.

(iii) $a^4+b^4+c^4=2(ab+ac+bc)^2$.

(i) $(ab+ac+bc)^2=a^2b^2+a^2c^2+b^2c^2+2ab(ac+bc)+2ac\times bc$ **by formula.**

$$=a^2b^2+a^2c^2+b^2c^2+2abc(a+b+c)$$
$$=a^2b^2+a^2c^2+b^2c^2+2abc\times 0$$
$$=a^2b^2+a^2c^2+b^2c^2.$$

(ii) Since $a+b+c=0$, $a+b=-c$. $\therefore (a+b)(a-b)=-c(a-b)$. $\therefore a^2-b^2=-ac+bc$. $\therefore a^2-bc=b^2-ac$.

Again, since $a+b+c=0$, $b+c=-a$. $\therefore (b+c)(b-c)=-a(b-c)$. $\therefore b^2-c^2=-ab+ac$. $\therefore b^2-ac=c^2-ab$.

But it has been shewn that $a^2-bc=b^2-ac$.

$\therefore \quad a^2-bc=b^2-ac=c^2-ab$.

(iii) Since $a+b+c=0$, $\quad \therefore (a+b+c)^2=0$

$\therefore \quad a^2+b^2+c^2=-2(ab+ac+bc)$

$\therefore \quad (a^2+b^2+c^2)^2=4(ab+ac+bc)^2$

$\therefore \quad a^4+b^4+c^4+2(a^2b^2+b^2c^2+a^2c^2)=4(ab+ac+bc)^2$

$\therefore \quad a^4+b^4+c^4+2(ab+ac+bc)^2=4(ab+ac+bc)^2$ by (i)

$\therefore \quad a^4+b^4+c^4=2(ab+ac+bc)^2$.

Ex. 2. If $3s=a+b+c$, shew that $(s-a)^3+(s-b)^3+(s-c)^3=3(s-a)(s-b)(s-c)$.

Let $s-a=x$, $s-b=y$ and $s-c=z$; then $x+y+z=s-a+s-b+s-c=3s-(a+b+c)=a+b+c-(a+b+c)=0$.

If $x+y+z=0$, then $x^3+y^3+z^3=3xyz$. $\therefore (s-a)^3+(s-b)^3+(s-c)^3=3(s-a)(s-b)(s-c)$.

Ex. 3. Prove that $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$, if $2s=a+b+c$.

$s=3s-2s=3s-(a+b+c)=s-a+s-b+s-c.$

$$\therefore s^3-(s-a)^3-(s-b)^3-(s-c)^3=(s-a+s-b+s-c)^3-(s-a)^3-(s-b)^3-(s-c)^3$$
$$=3\{s-a+s-b\}\{s-b+s-c\}\{s-c+s-a\}.$$

Because $(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)$

$$=3(2s-a-b)(2s-b-c)(2s-a-c)$$
$$=3(a+b+c-a-b)(a+b+c-b-c)(a+b+c-a-c)$$
$$=3abc.$$

Ex. 4. If $ab+ac+bc=1$, shew that $(1+a^2)(1+b^2)\times(1+c^2)=(a+b)^2(b+c)^2(c+a)^2=(a+b+c-abc)^2$.

Since $ab+ac+bc=1$. $\quad \therefore 1+a^2=a^2+ab+ac+bc=a(a+b)+c(a+b)=(a+b)(a+c)$.

Similarly $1+b^2=(b+c)(b+a)$; and $1+c^2=(c+a)(c+b)$
$\therefore (1+a^2)(1+b^2)(1+c^2)=(a+b)(b+c)(b+c)(b+a)(c+a)$
$\times(c+b)=(a+b)^2(b+c)^2(c+a)^2$
$=\{(a+b+c)(ab+ac+bc)-abc\}^2=(a+b+c-abc)^2.$

Ex. 5. If $a^3+b^3+c^3=3abc$, shew that
(i) $a+b+c=0$ or (ii) $a=b=c$.

Since $a^3+b^3+c^3=3abc$ $\therefore$ $a^3+b^3+c^3-3abc=0$
$\therefore$ $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)=0$
$\therefore$ $a+b+c=0$, or $a^2+b^2+c^2-ab-ac-bc=0$.

(If $x\times y=0$, then either $x=0$ or $y=0$).

(ii) $a^2+b^2+c^2-ab-ac-bc=0$
$\therefore$ $2a^2+2b^2+2c^2-2ab-2ac-2bc=0$
$\therefore$ $\{(a-b)^2+(b-c)^2+(c-a)^2\}=0.$

Each of $(a-b)^2$, $(b-c)^2$, $(c-a)^2$ is positive. **The sum of 3 positive quantities cannot be equal to zero unless each is equal to zero.**

Since $(a-b)^2=0$, $a-b=0$ $\therefore$ $a=b$
$(b-c)^2=0$, $b-c=0$ $\therefore$ $b=c$
$(c-a)^2=0$, $c-a=0$ $\therefore$ $c=a$ $\therefore$ $a=b=c$.

Ex. 6. If $x+\frac{1}{x}=1$, shew that $x^2+\frac{1}{x^2}=x^4+\frac{1}{x^4}=x^8+\frac{1}{x^8}$
$=x^{16}+\frac{1}{x^{16}}=$&c.

$$x^2+\frac{1}{x^2}=x^2+\frac{1}{x^2}+2-2=\left(x+\frac{1}{x}\right)^2-2=1-2=-1.$$

$$x^4+\frac{1}{x^4}=x^4+\frac{1}{x^4}+2-2=\left(x^2+\frac{1}{x^2}\right)^2-2=(-1)^2-2=-1.$$

$$x^8+\frac{1}{x^8}=x^8+\frac{1}{x^8}+2-2=\left(x^4+\frac{1}{x^4}\right)^2-2=(-1)^2-2=-1.$$

Similarly it may be shewn that $x^{16}+\frac{1}{x^{16}}=-1$, &c.

Ex. 7. Express $(a^2+b^2)(c^2+d^2)$ as the *sum* of two perfect squares.

$(a^2+b^2)(c^2+d^2)=a^2c^2+a^2d^2+b^2c^2+b^2d^2$
$=a^2c^2+b^2d^2+a^2d^2+b^2c^2+2abcd-2abcd$
$=(a^2c^2+b^2d^2+2abcd)+(a^2d^2+b^2c^2-2abcd)$
$=(ac+bd)^2+(ad-bc)^2$ [or $(ac-bd)^2+(ad+bc)^2$].

Ex. 8. Shew that $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square.

$$\begin{aligned}&(x+1)(x+2)(x+3)(x+4)+1\\&=\{(x+1)(x+4)\}\ \{(x+2)(x+3)\}+1\\&=(x^2+5x+4)(x^2+5x+6)+1\\&=(y+4)(y+6)+1\ [\text{putting } y \text{ for } (x^2+5x)]\\&=y^2+10y+24+1=y^2+10y+25=(y+5)^2\\&=(x^2+5x+5)^2.\end{aligned}$$

Ex. 9. Express $(2a+3b+4c)(4a+3b+2c)$ as the *difference* of two perfect squares.

Note. $ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2.$

Let $2a+3b+4c=x+y$; and $4a+3b+2c=x-y$
then $6a+6b+6c=2x$. $\therefore\ x=3(a+b+c)$
and $-2a+2c=2y$. $\therefore\ y=c-a$.
But $(x+y)(x-y)=x^2-y^2$.
$\therefore\ (2a+3b+4c)(4a+3b+2c)=\{3(a+b+c)\}^2-(c-a)^2$.

Ex. 10. If $x=a^2-bc$, $y=b^2-ac$ and $z=c^2-ab$, shew that $(a+b+c)(x+y+z)=ax+by+cz$

$$\begin{aligned}&(a+b+c)(x+y+z)=(a+b+c)(a^2-bc+b^2-ac+c^2-ab)\\&=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)\\&=a^3+b^3+c^3-3abc=(a^3-abc)+(b^3-abc)+(c^3-abc)\\&=a(a^2-bc)+b(b^2-ac)+c(c^2-ab)=ax+by+cz.\end{aligned}$$

EXERCISE 30.

Shew that—

1. $(a-b)^2-(b-c)(c-a)=(b-c)^2-(c-a)(a-b)=(c-a)^2-(a-b)(b-c)$.

2. $(a-b)^4+(b-c)^4+(c-a)^4=2\{(a-b)(b-c)+(b-c)\times(c-a)+(c-a)(a-b)\}^2$.

3. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$.

4. $(3a-2b-c)^3+(3b-2c-a)^3+(3c-2a-b)^3$
$=3(3a-2b-c)(3b-2c-a)(3c-2a-b)$.

5. $\dfrac{(a^3-b^3)^3+(b^3-c^3)^3+(c^3-a^3)^3}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ac+a^2)}=3(a-b)(b-c)\times(c-a)$.

If $a+b+c=0$, shew that—

6. $a^2+ab+b^2=b^2+bc+c^2=c^2+ac+a^2$.

7. $(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a)$.

8. $a(b+c)^2+b(c+a)^2+c(a+b)^2=3abc$.

9. $\dfrac{a^2+b^2+c^2}{a^3+b^3+c^3}=-\dfrac{2}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

10. $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^2=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$.

11. $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\times\dfrac{a^2+b^2+c^2}{2}$.

12. $\dfrac{a^2+b^2+c^2}{2}\times\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\times\dfrac{a^4+b^4+c^4}{4}$.

If $2s=a+b+c$, shew that—

13. $s^2+(s-a)^2+(s-b)^2+(s-c)^2=(a^2+b^2+c^2)$.

14. $4a^2b^2-(a^2+b^2-c^2)^2=16s(s-a)(s-b)(s-c)$.

15. $a(s-b)(s-c)+b(s-a)(s-c)+c(s-a)(s-b)+2(s-a)\times(s-b)(s-c)=abc$.

16. $(s-a)^2+(s-b)^2+(s-c)^2+2(ab+ac+bc)=3s^2$.

17. If $3s=2(a+b+c)$, then (i) $(2a-s)^3+(2b-s)^3+(2c-s)^3=3(2a-s)(2b-s)(2c-s)$; (ii) $(s-a-b)^3+(s-b-c)^3+(s-c-a)^3=3(s-a-b)(s-b-c)(s-c-a)$.

18. Prove that $(s-a)^3+(s-b)^3+3(s-a)(s-b)c=c^3$, if $2s=a+b+c$.

19. Shew that $(x-a)(x-b)(a-b)+(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)=(a-b)(b-c)(a-c)$.

20. If $x^2-yz=a^2$, $y^2-xz=b^2$, $z^2-xy=c^2$, prove that $(a^2x+b^2y+c^2z)=(x+y+z)(a^2+b^2+c^2)$.

21. If $x+y=a$, and $xy=b^2$, shew that $x^3+y^3=a^3-3b^2a$ and $x^4+y^4=a^4-4a^2b^2+2b^4$.

22. Shew that $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2=0$, if $a+b=c$.

23. If $x+\dfrac{1}{x}=a$, shew that $x^3+\dfrac{1}{x^3}+x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=(a+1)\times(a^2-2)$.

24. If $ab=xy$ and $ma+\dfrac{b}{m}=x+y$, shew that $x^3+y^3=m^3a^3+\dfrac{b^3}{m^3}$.

25. If $a+b+c=ab+ac+bc$, $(a+b)(b+c)(c+a)+abc$ is a perfect square.

26. Shew that (i) $(x+2)(x+4)(x+6)(x+8)+16$ and (ii) $(x+a)(x+2a)(x+3a)(x+4a)+a^4$ are perfect squares.

27. Express $(a+b+c-d)(a+b-c+d)$ as the *difference* of two squares.

28. Shew that $(x^2+y^2+z^2)(a^2+b^2+c^2) = (ax+by+cz)^2 + (ay-bx)^2+(cx-az)^2+(cy-bz)^2$.

29. Shew that $a^5+b^5+c^5+5abc\,(ab+ac+bc)=0$, if $a+b+c=0$.

30. Shew that $4^{2n-1}+4^{n+1}-4^n+9$ is a perfect square.

31. Shew that (i) $3(2a-b-c)(c-a)=$ difference of two squares; (ii) $2(x^2+y^2+z^2-xy-yz-zx)=$ the sum of 3 squares.

Prove that—

32. $$(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2 + (a+b-c-d)^2=4(a^2+b^2+c^2+d^2).$$

33. $$a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3=a^2(b^3-c^3) + b^2(c^3-a^3)+c^2(a^3-b^3).$$

34. $$(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a) =2(a^3+b^3+c^3-3abc).$$

35. $$abc(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc+(a+b)(b+c)(c+a).$$

36. $(a+b)^5-a^5-b^5=5ab(a+b)(a^2+ab+b^2)$.

37. If $x+y=a$, $xy=b$, shew that $(1+x^2)(1+y^2)=a^2+(1-b)^2$.

38. $a^2(b+c)+b^2(c+a)+c^2(a+b)-a^3-b^3-c^3-2abc =(b+c-a)(c+a-b)(a+b-c)$.

39. $(a^2-bc)(b-c)+(b^2-ac)(c-a)+(c^2-ab)(a-b)=0$.

40. $(x^2-yz)^3+(y^2-xz)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)\times(z^2-xy)=(x^3+y^3+z^3-3xyz)^2=(x^2+2yz)^3+(y^2+2xz)^3+(z^2+2xy)^3-3(x^2+2yz)(y^2+2xz)(z^2+2xy)$.

41. If $a=y+z-x$, $b=z+x-y$, $c=x+y-z$, then $a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz)$.

42. Shew that $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2+(b+c-a)(c+a-b)(a+b-c)=4abc$.

CHAPTER XI.

HIGHEST COMMON FACTOR.

80. Definitions—One quantity is said to be a *measure* of another, when the former is contained in the latter a certain number of times exactly; thus $4a$ is a measure of $12a$, since $4a$ is contained in $12a$ exactly three times.

If two or more algebraical expressions be arranged according to descending powers of some common letter, then the factor of the highest dimensions with respect to that letter that divides each of them without remainder is called their *greatest common measure*.

The term *greatest common measure* is not appropriate in Algebra; because when numerical values are assigned to the letters contained in the expressions, the numerical value of the G.C.M. is not necessarily the Arithmetic G.C.M., of the resulting numbers. Thus $x-4$ is the G.C.M. of $(x-4)(3x+2)$ and $(x-4)(x+18)$; when $x=8$, the G.C.M. $x-4$ becomes 4, and the expressions become 104 and 104 whose Arithmetic G.C.M. is not 4, but 104.

The term *Highest Common Factor or Divisor* is therefore to be preferred.

81. The H.C.F. of simple expressions.

Rule.—"Find the H.C.F. of the numerical co-efficients, and annex all the letters which are common to all the expressions, and raise each letter to the lowest power to which it occurs in any of them."

Ex. Find the H.C.F. of $5x^3yz^3$, $15x^3y^2z$ and $20x^2y^2z^2$.

The H.C.F. of 5, 15 and 20 is 5; the letters common to the expressions are x, y and z; the lowest powers to which they occur are x^2, y and z; therefore the H.C.F. is $5x^2yz$.

EXERCISE 31.

Find the H.C.F. of :—

1. $2x^2b^3$ and x^3b^2.
2. $15x^3b$ and $20x^2z^3$.
3. $9ab^2c^3$ and $24a^3b^4$.
4. $18p^2q^4$ and $45p^5q^3$.
5. $2x^3y^2$, $3x^2y^3$ and $4x^4y^4z$.
6. $17a^5b^2c^3$, $51a^4b^3c^4$ and $68a^6b^3c^4$.
7. $36a^2b^3c^4x^5$, $54a^5c^4x^4$ and $81a^4b^3c^5$.
8. $14a^{m+n}b^{p+q}$, $70a^{m+1}b^{p+q-1}$ and $110a^{m+2}b^{p+q-2}$.

82. H.C.F. of Compound Expressions which can be readily resolved into factors.

Ex. 1. Find the H.C.F. of a^2-b^2, $a^2-2ab+b^2$ and a^3-b^3.

Here $a^2-b^2=(a+b)(a-b)$, $a^2-2ab+b^2=(a-b)^2$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$.

We see at once that $a-b$ is the only common factor; therefore the H.C.F. is $a-b$.

Ex. 2. Find the H.C.F. of x^2+4x+3, x^2-2x-3 and x^2+2x+1.

Here $x^2+4x+3=(x+1)(x+3)$, $x^2-2x-3=(x+1)(x-3)$ and $x^2+2x+1=(x+1)^2$. Therefore the H.C.F. is $x+1$, because it is the only common factor.

EXERCISE 32.

Find the H.C.F. of :—

1. a^3+b^3, a^2-ab+b^2 and $a^4+a^2b^2+b^4$.
2. a^3+1, a^2+4a+3 and a^2+6a+5.
3. x^2-1, x^2+3x+2 and x^3+3x^2+3x+1.
4. $x^2+7x-18$, x^3-8 and x^2-4.
5. $x^2-(b+c)x+bc$, $x^2-(a+b)x+ab$ and $x^2-2bx+b^2$.
6. $x^2-7xy+6y^2$ and x^3-xy^2.
7. x^4+4 and x^2+2x+2.
8. x^3-x, $2x^2-4x+2$ and x^3+x^2-2x.
9. $4x^3+12x^2+9x$ and $4x^2-2x-12$.
10. $a^2+3a-10$ and $a^3-a^2-14a+24$.
11. $a^2+ab+ac+bc$, $a^2+2ac+c^2$ and a^3+c^3.
12. $x^3+y^3+z^3-3xyz$ and $x^2+2xy+y^2-z^2$.

83. H.C.F. of two Compound Expressions whose factors cannot be readily found.

Rule.—"Arrange the expressions according to descending or ascending powers of some common letter; take as divisor that one whose degree in the common letter is not higher than the degree of the other, and the remaining one as the dividend; perform the division till you get a remainder of lower degree than the divisor; consider this remainder as a new divisor, and the last divisor as a new dividend; then consider the new remainder arising from this division as another new divisor, and the last divisor as another new dividend. Continue this process till there is no remainder; the last divisor is the H.C.F. required."

84. The truth of the rule depends upon the following principles :—

(1) If p divides A, then it will divide mA. For since p divides A, we may suppose $A=xp$; then $mA=mxp$; thus p divides mA.

(2) If p divides A and B, then it will divide $mA \pm nB$. For since p divides A and B, we may suppose $A=xp$, and $B=yp$; then $mA \pm nB = mxp \pm nyp = (mx \pm ny)p$. Thus p divides $mA \pm nB$.

85. Proof of the rule given in Article 83. Let A and B be two expressions arranged according to descending powers of some common letter, and let the index of the highest power of that letter in A be not less than the index of the highest power of that letter in B.

$$\begin{array}{l} B)A(p \\ \underline{pB} \\ C)B(q \\ \underline{qC} \\ D)C(r \\ \underline{rD} \\ 0 \end{array}$$

Divide A by B; let p be the quotient and C the remainder. Divide B by C; let q be the quotient and D the remainder. Divide C by D, and suppose that there is no remainder, and let r denote the quotient.

Thus we have the following results : $A=pB+C$, $B=qC+D$ and $C=rD$.

First, we shall show that D is a common factor of A and B.

Since $C=rD$, D divides C.

$\therefore$ D divides qC and also $qC+D$. (Art. 84).

That is, D divides B; for $B=qC+D$.

Again, since D divides B and C, it divides $pB+C$. (Art. 84).

That is, D divides A; for $A=pB+C$.

Therefore D divides A and B; that is, D is a common factor of A and B.

Secondly, we shall show that D is the H.C.F. of A and B.

For, every expression which divides A and B divides $A-pB$, or C. (Art. 84).

Thus every expression which is a common factor of A and B is a common factor of B and C.

Again, every expression which divides B and C divides $B-qC$ or D. (Art. 84).

Thus every expression which divides B and C divides C and D.

Therefore every expression which is a common factor of A and B is a factor of D.

But no expression higher than D can divide D exactly. Therefore D is the H.C.F. of A and B.

Note.—Every common factor of A and B is a factor of their H.C.F.

86. The process of finding the H.C.F. is often simplified by the following artifices:—

(1) We may remove any factor of either of the expressions which is not a factor of the other.

(2) We may multiply either of the expressions by any quantity which does not introduce a common factor.

(3) Simple factors of the expressions may be removed; the H.C.F. of the expressions is the product of the H.C.F. of these factors and the H.C F. of the quotients.

(4) We may treat the divisor and the dividend, at any stage of the process, in the same way as the original expressions.

Ex. 1. Find the H.C.F. of $9a^4+9a^3+9a^2-27a$ and $6a^5+18a^4+30a^3+18a^2$.

These expressions are equal to $9a(a^3+a^2+a-3)$ and $6a^2(a^3+3a^2+5a+3)$ respectively.

By the third rule of this Article, we may remove the factors $9a$ and $6a^2$ whose H C.F. is $3a$; we must now find the H.C.F. of a^3+a^2+a-3 and a^3+3a^2+5a+3, and multiply it by $3a$.

$$\begin{array}{rll}
a^3+a^2+a-3) & a^3+3a^2+5a+3\,(1 & \\
 & a^3+a^2+a-3 & \\
 & 2\,|\,\overline{2a^2+4a+6} & \text{(Remove the factor 2 which is not} \\
a^2+2a+3) & a^3+a^2+a-3\,(a-1 & \text{a factor of the} \\
 & a^3+2a^2+3a & \text{Dividend).} \\
 & -a^2-2a-3 & \\
 & -a^2-2a-3 & \\
\end{array}$$

$\therefore$ a^2+2a+3 is the H.C.F. of a^3+a^2+a-3 and a^3+3a^2+5a+3.

Hence the H.C.F. of the given expressions $=3a(a^2+2a+3)$.

Ex. 2. Find the H.C.F. of $2x^3-15x+14$ and $x^4-15x^2+28x-12$.

To avoid fractions in the quotient we multiply the latter by 2, since 2 is not a factor of the other expression.

$$\begin{array}{rll}
 & x^4-15x^2+28x-12 & \\
 & 2 & \\
2x^3-15x+14) & \overline{2x^4-30x^2+56x-24}\,(x & \\
 & 2x^4-15x^2+14x & \\
 & -3\,|\,\overline{-15x^2+42x-24} & \text{(Divide by}-3\text{ which is not} \\
 & 5x^2-14x+8 & \text{a factor of the divisor).} \\
\end{array}$$

Multiply each term of $2x^3-15x+14$ by 5 which is not a factor of $5x^2-14x+8$.

$$
\begin{array}{l}
5x^2-14x+8)\,10x^3-75x+70\,(2x \\
\qquad\qquad\qquad 10x^3-28x^2+16x \\
\qquad\qquad\quad 7\,|\,\overline{28x^2-91x+70} \qquad \text{(Divide by 7).} \\
\qquad\qquad\qquad \overline{4x^2-13x+10.}
\end{array}
$$

Multiply $5x^2-14x+8$ by 4 to avoid fractions in the quotient

$$
\begin{array}{l}
4x^2-13x+10)\,20x^2-56x+32\,(5 \\
\qquad\qquad\qquad 20x^2-65x+50 \\
\qquad\qquad\quad 9\,|\,\overline{9x-18} \qquad\qquad \text{(Divide by 9).} \\
\qquad\qquad x-2)\,4x^2-13x+10\,(4x-5 \\
\qquad\qquad\qquad 4x^2-8x \\
\qquad\qquad\qquad\quad \overline{-5x+10} \\
\qquad\qquad\qquad\quad -5x+10
\end{array}
$$

Therefore the H.C.F. is $x-2$.

Ex. 3. Find the H.C.F. of $4x^3-9x^2+6x-1$ and $6x^3-7x^2+1$.

In this case, it is advantageous to arrange the expressions according to ascending powers of x.

$$
\begin{array}{l}
1-7x^2+6x^3)-1+6x-9x^2+4x^3\,(-1+3 \\
\qquad\qquad\qquad -1+7x^2-6x^3 \\
\qquad\qquad\quad 2x\,|\,\overline{6x-16x^2+6x^3+4x^4} \qquad \text{(Divide by } 2x\text{).} \\
\qquad\qquad\qquad \overline{3-8x+3x^2+2x^3} \\
\qquad\qquad\qquad 3-21x^2+18x^3 \\
\qquad\qquad -8x\,|\,\overline{-8x+24x^2-16x^3} \qquad \text{(Divide by } -8x\text{).} \\
\qquad\qquad\qquad 1-3x+2x^2)\,1-7x^2+6x^3\,(1+3x \\
\qquad\qquad\qquad\qquad\qquad 1-3x+2x^2 \\
\qquad\qquad\qquad\qquad\qquad \overline{3x-9x^2+6x^3} \\
\qquad\qquad\qquad\qquad\qquad 3x-9x^2+6x^3
\end{array}
$$

Therefore the H.C.F. is $1-3x+2x^2$.

87. H.C.F. of more than two expressions.

If we have *three* expressions A, B, C, first find the H.C.F. of two of them, A and B; let D be the H.C.F.; then the H.C.F. of D and C is the H.C.F. of A, B and C.

In the same way, we may find the H.C.F. of *four* algebraical expressions.

Ex. Find the H.C.F. of a^2-3a+2, a^2-4a+3 and $a^3-6a^2+11a-6$.

The H. C. F. of the first two is $a-1$, which is found to be a factor of the last; hence $a-1$ is the H.C.F. of the three expressions.

EXERCISE 33.

Find the H.C.F. of :—

1. x^3+x^2+x-3 and x^3+3x^2+5x+3.
2. $x^4-3x^3+2x^2+x-1$ and x^3-x^2-2x+2.
3. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
4. $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$.
5. $6x^4+x^3-x$ and $4x^3-6x^2-4x+3$.
6. $3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$.
7. x^5+11x^3-54 and $x^5+11x+12$.
8. $11x^4+24x^3+125$ and $x^4+24x+55$.
9. $1-4x^3+3x^4$ and $1+x-x^2-5x^3+4x^4$.
10. $12a^2x^4+120a^4x^2-132a^5x$ and $3a^2x^8-27a^3x^7+39a^4x^6-15a^7x^3$.
11. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$.
12. $25x^4+5x^3-x-1$ and $20x^4+x^2-1$.
13. x^3-7x+6 and $6x^3-7x^2+1$.
14. x^5+2x^2-x+1 and x^5+x-1.
15. $5x^4-x^3-x^2-5x+2$ and $5x^4-x^3-7x^2+5x-2$.
16. $ax^3+x^2(b-a)-x(a^2+b+ab)+a(a+b)$ and $bx^3-x^2(a+b-ab)-ax(a+b-1)+a^2$.
17. $1+a^2+a^4$ and $1+2a+a^2-a^4$.
18. $3x^3-7x^2-18x-8$ and $2x^3-3x^2-17x-12$.
19. $2x^3+9x^2a+xa^2-12a^3$ and $2x^4+15x^3a+40x^2a^2+45xa^3+18a^4$.
20. $x^5+2x^4-5x^2-7x+3$ and $3x^6-3x^4-18x^3+x^2+2x+3$.
21. $4x^5-8x^3a^2+28x^2a^3-24xa^4+24a^5$ and $6x^4+24x^3a-12x^2a^2-24xa^3+96a^4$.
22. $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$ and $x^4-(a-b)x^3+(a-b)b^2x-b^4$.
23. $3x^2+(4a-2b)x-2ab+a^2$ and $x^3+(2a-b)x^2-(2ab-a^2)x-a^2b$.

24. $\frac{x^2}{3}+\frac{11x}{6}\sqrt{x+1}-x-1$ and $x^2-\frac{x}{4}-\frac{1}{4}$.

25. $a^3+3a^2b+3ab^2+b^3$, $a^3+a^2b+ab^2+b^3$ and $a^3+a^2b-ab^2-b^3$.

26. $x^3-9x^2+26x-24$, $x^3-10x^2+31x-30$, and $x^3-11x^2+38x-40$.

27. $x^2-3x-70$, $x^3-39x+70$ and $x^3-48x+7$.

28. $1-5x^2+4x^4$, $1+2x-3x^3-6x^4$ and $1+2x-4x^3-8x^4$.

88. To find the H.C.F. of two expressions by destroying the highest and lowest terms alternately.

Ex. 1. Find the H.C.F. of a^4+2a^3+2a+1 and a^3-a^2-7a+3.

Let $a^4+2a^3+2a+1=A$ and $a^3-a^2-7a+3=B$.

The H.C.F. of A and B is a factor of $A-B\times a=a^4+2a^3+2a+1-a(a^3-a^2-7a+3)=3a^3+7a^2-a-1$ (suppose this $=C$).

The H.C.F. of A and B is a factor of $C-3B=3a^3+7a^2-a-1-3(a^3-a^2-7a+3)=10(a^2+2a-1)$.

Again the H.C.F. of A and B is a factor of $B+3C=a^3-a^2-7a+3+3(3a^3+7a^2-a-1)=10a^3+20a^2-10a=10a(a^2+2a-1)$.

We have shown that the H.C.F. of A and B is a factor of $10(a^2+2a-1)$ and of $10a(a^2+2a-1)$.

Since $10a$ is not common to them, a^2+2a-1 is their H.C.F.

Ex. 2. Find the H.C.F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

If D be their H.C.F., then D is a factor of $4x^5+11x^4+81-2(2x^5-11x^2-9)$, *i.e.*, of $11(x^4+2x^2+9)$, *i.e.*, of x^4+2x^2+9...(i).

Again, D is a factor of $9(2x^5-11x^2-9)+(4x^5+11x^4+81)$, *i.e.*, of $11x^2(2x^3+x^2-9)$, *i.e.*, of $(2x^3+x^2-9)$....(ii). Therefore D is a factor of (i) and (ii). $\therefore$ it is a factor of their sum, *i.e.*, of $x^2(x^2+2x+3)$, *i.e.*, of x^2+2x+3.

Since x^2+2x+3 cannot be further resolved, $D=x^2+2x+3$.

89. Miscellaneous examples.

Ex. 1. What value of a will make
$2(a^2+a)x^2+(11a-2)x+4$ and $2(a^3+a^2)x^3+(11a^2-2a)x^2+(a^2+5a)x+5a-1$ have a common factor?

Divide the one expression by the other as in finding their H.C.F.

$$\begin{array}{l|l|l}
2(a^2+a)x^2+ & 2(a^3+a^2)x^3+(11a^2-2a)x^2+(a^2+5a)x+5a-1 & ax \\
(11a-2)x+4 & \underline{2(a^3+a^2)x^3+(11a^2-2a)x^2+4ax} & \\
 & \qquad\qquad (a^2+a)x+5a-1 &
\end{array}$$

$$\begin{array}{l|l|l}
(a^2+a)x+5a-1 & 2(a^2+a)x^2+(11a-2)x+4 & 2x \\
 & \underline{2(a^2+a)x^2+10ax-2x} & \\
 & \qquad\qquad ax+4 &
\end{array}$$

$$\begin{array}{l|l|l}
ax+4 & (a^2+a)x+5a-1 & a+1 \\
 & \underline{(a^2+a)x+4a+4} & \\
 & \qquad\qquad a-5 &
\end{array}$$

In order that the expressions may have a common factor, the remainder $a-5$ must $=0$ $\therefore a=5$.

Ex. 2. If $x+p$ be a common factor of ax^2+bx+c and cx^2+bx+a, find the value of p.

Dividing ax^2+bx+c by $x+p$, the remainder is ap^2-bp+c. Since $x+p$ is a factor of ax^2+bx+c, this remainder $ap^2-bp+c=0$...(i).

Again, dividing cx^2+bx+a by $x+p$, the remainder is cp^2-bp+a. For the same reason, $cp^2-bp+a=0$......(ii).

From (i) and (ii), we have $p^2(a-c)+c-a=0$.

$\therefore p^2=\dfrac{a-c}{a-c}=1$ $\therefore p=\pm 1$. $\therefore x\pm 1$ is a factor of each of the expressions ax^2+bx+c and cx^2+bx+a.

$\therefore a+b+c=0$, or $a+c=b$. . . . (Articles 70 and 71).

Ex. 3. Find the condition that ax^2+bx+c and dx^2+ex+f should have a common factor of the form $x+p$.

The common factor of ax^2+bx+c and dx^2+ex+f is a factor of $d(ax^2+bx+c)-a(dx^2+ex+f)$, *i.e.*, of $x(bd-ae)+cd-af$...(i).

Again, the common factor of ax^2+bx+c and dx^2+ex+f is a factor of $f(ax^2+bx+c)-c(dx^2+ex+f)$, *i.e.*, of $x^2(af-cd)+x(bf-ec)$.

Rejecting the factor x which is not a common factor of the given expressions, we have the common factor of the given expressions as a factor of $x(af-cd)+bf-ec$ . . . . (ii).

It is clear that if the given expressions have a factor of the form $x+p$, it must be a factor of $x(bd-ae)+cd-af$ and $x(af-cd)+bf-ec$, i.e., of $(bd-ae)\left(x+\dfrac{cd-af}{bd-ae}\right)$

and $(af-cd)\left(x+\dfrac{bf-ec}{af-cd}\right)$.

Therefore $x+p=\left(x+\dfrac{cd-af}{bd-ae}\right)=x+\left(\dfrac{bf-ec}{af-cd}\right)$.

$\therefore \dfrac{cd-af}{bd-ae}=\dfrac{bf-ec}{af-cd}$, or $(cd-af)^2=(bf-ec)(ac-bd)$ which is the condition required.

EXERCISE 34.

1. If $x+a$ be the H.C.F. of x^2+px+q and $x^2+p'x+q'$, prove that $(p-p')a=q-q'$.

2. If x^2+px+q and $x^2+p'x+q'$ have a common factor of the form $x+a$, shew that it is also a factor of $px^2+(q-p')x-q'$.

3. If $x+a$ be a common factor of x^2+px+q and x^2+rx+s, find the value of a in terms of p, q, r and s.

4. If $x+a$ be the H.C.F. of x^2+px+q and x^2+qx+p, then either $a+1=0$ or $p=q$.

5. If x^3-px^2+qx-r and $x^3-p'x^2+q'x-r$ have a common factor of the form $x-a$, then $a=\dfrac{q-q'}{p-p'}$.

6. If the expressions $ax^3-(3a+b)cx^2+(a^3+bc^2)x+d$ and $bx^3+(a-b)cx^2+a(c^2-a^2)x-d$ have a common factor of the form px^2+qx+r, prove that this factor is an exact square.

7. Find the relation between a, b and c in order that ax^3+bx+c and cx^3+bx^2+a may have a common factor of the form x^2+x+1.

8. If $x+p$ be the H.C.F. of $x^2+ax+ab$ and $x^2+cx+bc$, shew that $p=b$.

9. If $x+p$ be the H.C.F. of x^2+ax+b and $x^2+2bx+3a$, shew that $p=\dfrac{b-3a}{a-2b}$.

10. If $x+a$ be the H.C.F. of x^2+mx+1 and x^2+nx+2, shew that $a=\dfrac{1}{n-m}$.

11. If ax^2+bx+c and $ax^2+(a+b)x+a$ have a common factor of the form $x+p$, shew that each of the expressions $(a-c)\times(b+c)$ and $(a-b)(c-a)$ is an exact square.

12. If $x+p$ be a common factor of x^2+bx+c and x^2+ax+d, shew that it is a factor of $bx^2+(c-a)x-d$.

13. If x^2-3x+a and $x^2+2x+6a$ have a common factor, find it.

14. What value of m will make $x^3-(m-6)x^2-2mx+24$ and $x^3-(m+3)x^2-(2m-45)x-30$ have a common factor?

15. For what value of a will $x^3-ax^2+19x-a-4$ and $x^3-(a+1)x^2+23x-a-7$ have a common factor?

16. If x^3+px+q and $x^3+p'x+q'$ have a common factor, then $(q'-q)^3+(p'-p)^2(pq'-p'q)=0$.

CHAPTER XII.

LOWEST COMMON MULTIPLE.

90. When one algebraic expression contains another as a factor, the former is called a *multiple* of the latter.

The *Lowest Common Multiple* of any number of algebraic expressions, arranged according to powers of some common letter is the expression of the *lowest* degree in that letter, which is divisible by each of the expressions, without remainder.

The term *least common multiple* is not appropriate in Algebra; for when numerical values are assigned to the letters of the expressions, the numerical value of their L.C.M. is not necessarily the L.C.M. of the values of the expressions. Therefore, the term *Lowest Common Multiple* should be preferred.

91. L.C.M. of Simple Expressions, and of such Compound Expressions as can be easily resolved into factors.

Rule.—" Find the L.C.M. of the numerical factors and after it write down all the factors that occur in the different expressions, raising each factor to the highest power to which it occurs in any of the expressions."

Ex. 1. Find the L.C.M. of $18a^3b^2c$, $6a^2b^3d$ and $7a^4d^5$.

The L.C.M. of 18, 6 and 7 is 126. The different factors that occur are a, b, c and d; the highest power to which they occur are a^4, b^3. c and d^5.

Therefore $126a^4b^3cd^5$ is the L.C.M.

Ex. 2. Find the L.C.M. of $7(x^2-y^2)^2$, $5(x-y)^2$ and $21(x^3-y^3)$.

The L.C.M. of 7, 5 and 21 is 105; and $(x^2-y^2)^2=(x+y)^2 \times(x-y)^2$; and $x^3-y^3=(x-y)(x^2+xy+y^2)$.

The different factors are $x+y$, $x-y$ and x^2+xy+y^2.

Their highest powers are $(x+y)^2$, $(x-y)^2$ and x^2+xy+y^2. Therefore the L.C.M. $=105(x+y)^2(x-y)^2(x^2+xy+y^2)$.

EXERCISE 35.

Find the L.C.M. of :—

1. $3x^2a$, $6x^2y^2z^3$ and $27x^4y^4z^4$.
2. $8a^3b^2c$, $12ab^3c^2$ and $20a^2bc^2$.

3. $x^3b - xb^3$ and $x^3b^2 + x^2b^3$.

4. $x^2 + 4x + 3$ and $x^2 - 2x - 15$.

5. $x^2 - a^2$, $x^3 - a^3$ and $x^4 - a^4$.

6. $x^2 + 5x + 4$, $x^2 + 2x - 8$ and $x^2 + 7x + 12$.

7. $x^2 + (a+b)x + ab$, $x^2 + (b+c)x + bc$ and $x^2 + (c+a)x + ca$.

8. $x^2 + (b-c)x - bc$, $x^2 + (c-a)x - ca$ and $x^2 + (a-b)x - ab$.

9. $2x^2 - x - 1$, $2x^2 + 3x + 1$ and $4x^4 - 5x^2 + 1$.

10. $(a^2 - b^2)^3$, $(a^3 - b^3)^2$ and $(a^4 - b^4)(a-b)^3$.

11. $a^2(b-c) + b^2(c-a) + c^2(a-b)$ and $(a-b)^3 + (b-c)^3 + (c-a)^3$.

12. $a^6 - b^6$, $a^4 + a^2b^2 + b^4$ and $(a+b+c)(ab+ac+bc) - abc$.

92. L.C.M. of two Compound Expressions.

Rule.—"Divide either of the expressions by their H.C.F., and multiply the quotient by the other." or "Divide the product of the expressions by their H.C.F., and the quotient is their L.C.M."

Let A and B be the two given expressions, and D their H.C.F.; then $A = pD$ and $B = qD$, where p and q have no common factor, since D is the highest common factor of A and B.

The L.C.M. of pD and qD is clearly pqD; but

$$pqD = p \times qD = pB = \frac{A}{D} \times B = \frac{A \times B}{D}.$$

$$\text{or } = q \times pD = qA = \frac{B}{D} \times A = \frac{A \times B}{D}.$$

This proves the rule.

Note.—Since the L.C.M. of A and $B = (A \times B)$ divided by their H.C.F., it follows that *the product of A and B = their H.C.F. × their L.C.M.*

Ex. 1. Find the L.C.M. of $x^3 - 3x^2 + 3x - 1$ and $x^3 - x^2 - x + 1$.

First expression $= (x-1)^3$.

Second expression $= x^2(x-1) - (x-1) = (x-1)(x^2-1)$

$= (x-1)^2(x+1)$. Therefore their H.C.F. is $(x-1)^2$

Hence the L.C.M. $= \dfrac{(x-1)^3(x-1)^2(x+1)}{(x-1)^2} = (x-1)^3(x+1)$.

Ex. 2. Find the L.C.M. of $x^3 + 3x^2 - 4x - 12$ and $x^3 + 2x^2 - x - 2$.

First expression $= x^2(x+3) - 4(x+3) = (x^2-4)(x+3)$

$= (x+2)(x-2)(x+3)$.

Second expression $= x^2(x+2)-(x+2)=(x^2-1)(x+2)$
$=(x+1)(x-1)(x+2)$. Their H C F. is $x+2$.

Hence the L.C.M. $=\dfrac{(x^2-4)(x+3)(x^2-1)(x+2)}{x+2}$
$=(x^2-4)(x+3)(x^2-1)$.

93. Every Common Multiple of two algebraical expressions is a Multiple of their Lowest Common Multiple.

Let A and B be the two expressions, and M their L.C.M., and let N denote any other common multiple. Suppose, if possible, that when N is divided by M there is a remainder R; q being the quotient. Then $R=N-qM$.

Since A and B are factors of M and N, therefore they are also factors of $N-qM$, *i.e.*, of R. (Art. 84).

Thus R is a multiple of A and B; but R is of *lower* dimensions than M, which, by supposition, is their L.C.M. This is absurd; hence there can be no remainder R; that is, N is a multiple of M.

94. L C.M. of three or more Compound Expressions.

Find the L C.M. of two of them (A and B). If M be this L.C.M., then the L.C.M. of M and C is the L.C M. of A, B, and C. If N be this L.C.M., then the L C.M. of N and D is the L.C.M. of A, B, C and D; and so on.

Note. We can easily deduce from the foregoing Articles that

(1) **The L.C.M. of two expressions is the H.C.F. of all their common multiples.**

(2) **The H.C.F. of two expressions is the L C.M. of all their common factors.**

95. Miscellaneous Examples

Ex 1. If $x+p$ be the H.C.F. of x^2+ax+b and x^2+cx+d, shew that their L.C.M. $=x^3+(a+c-p)x^2+(ac-p^2)x+p(a-p)\times(c-p)$.

Dividing x^2+ax+b by $x+p$, the quotient is $x+a-p$ and the remainder is p^2-ap+b.

Since $x+p$ is a factor of x^2+ax+b, the remainder p^2-ap+b must $=0$. $\therefore b=ap-p^2=p(a-p)$.

Again, dividing x^2+cx+d by $x+p$, the quotient is $x+c-p$ and the remainder is p^2-cp+d.

For the same reason, $p^2-cp+d=0$. $\therefore d=cp-p^2=p(c-p)$.

We have $x^2+ax+b=(x+p)(x+a-p)$ and $x^2+cx+d=(x+p)(x+c-p)$. The L.C.M. of the two expressions

$$=\frac{(x^2+ax+b)(x^2+cx+d)}{x+p}=(x+a-p)(x^2+cx+d)$$
$$=(x+a-p)(x^2+cx+cp-p^2)\ \because\ d=p(c-p)$$
$$=x^3+(a+c-p)x^2+(ac-p^2)x+p(a-p)(c-p).$$

Ex. 2. The H.C.F. of two expressions is $x-4$; and their L.C.M. is $x^3-9x^2+26x-24$ Find the expressions.

The L.C.M. $=x^3-9x^2+26x-24=(x-2)(x-3)(x-4)$.

The H.C.F $=x-4$. $\therefore$ the product of the expressions $=$ H.C.F. $\times$ L.C.M. $=(x-4)(x-2)(x-3)(x-4)$.

It is plain that each expression must contain $x-4$ as a factor, and that the other factors of the expressions must have no common factor. Therefore the expressions are $(x-4)(x-2)$ and $(x-4)(x-3)$; or $x-4$ and $(x-4)(x-2)(x-3)$.

EXERCISE 38.

(The student may, with advantage, find the L.C.M. of the expressions in Exercise 33).

Find the L.C.M. of:—

1. $6a^3-11a^2+5a-3$ and $9a^3-9a^2+5a-2$.
2. $x^3-9x^2+26x-24$ and $x^3-10x^2+31x-30$.
3. $2x^3+(2a-3b)x^2-(2b^2+3ab)x+3b^3$ and $2x^2-(3b-2c)\times x-3bc$.
4. $4x^4+4x^3+7x^2+11x+4$ and $6x^4+7x^3+4x^2+5x+2$.
5. $4a^4+8a^3+21a^2+18a+27$ and $3a^4+6a^3+17a^2+16a+24$.
6. $6x^2-11x+3$, $4x^2-4x-3$ and $6x^2+25x-9$.
7. $6a^2-19a+10$, $12a^2-11a+2$ and $8a^2+10a-3$.
8. $a^4-3a+20$, $a^3-5a+12$ and $3a^3-5a^2+16$.
9. x^5-2x^4-x-1 and x^5+2x^3+x+1.
10. $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$.
11. If the H.C F. and L.C M. of A and B be x and y respectively, and if $x+y=ma+\frac{b}{m}$; shew that $x^3+y^3=m^3a^3+\frac{y^3}{m^3}$.
12. The L.C.M. of two quantities is $x^4-5a^2x^2+4a^4$, and their H.C.F. is x^2-a^2; one of the quantities is $x^3-2ax^2-a^2x+2a^3$; find the other.

13. If $x+d$ be the H.C.F. of x^2+ax+b and $x^2+a'x+b'$, prove that their L.C.M. is $x^3+(a+a'-d)x^2+(aa'-d^2)x+(a-d)(a'-d)d$.

14. If x be the H.C.F. of a and b, and y be their L.C.M., shew that $a^2+b^2+x^2+y^2$ is the sum of two squares.

15. If $x+p$ be the H.C.F. of x^2+ax+b and x^2+cx+b, their L.C.M. will be $x^3+x^2(2a-p)+x\left\{\dfrac{b(a+p)}{p}\right\}+\dfrac{b^2}{p}$.

16. If x^2+nx+1 and x^3+nx^2+1 have a common factor, determine n; and then shew that the L.C.M. of these quantities is $x^4-3x^3+2x^2+x+1$.

17. If the H.C.F. of two expressions be $x-5$ and their L.C.M. be $x^3-13x^2+52x-60$, find the expressions.

18. If the H.C.F. of two expressions be $x+4$ and their L.C.M. be $x^3-4x^2-17x+60$, find the expressions.

19. If $x+1$ be the H.C.F. of two expressions and $x^3+6x^2+11x+6$ be their L.C.M., find the expressions.

20. If $x+d$ be the H.C.F. of $x^2+ax+ad$ and $x^2+cx+dc$, their L.C.M. will be $x^3+x^2(a+c)+acx$.

21. If x^2+ax+b and x^2+px+q have a common factor of the first degree, then their L.C.M. $=x^3+\dfrac{ab-pq}{b-q}x^2$

$$+\left\{ap-\left(\frac{b-q}{a-p}\right)^2\right\}x+bq\,\frac{a-p}{b-q}.$$

22. If H_1, H_2, H_3, be the H.C.F. of A and B, B and C, C and A respectively, then the L.C.M. of A, B and $C=\dfrac{ABC}{H_1H_2H_3}\times$ H.C.F. of A, B and C.

23. If the H.C.F. of $ax^2+2bx+c$ and $bx^2+2ax+c$ is of the form $x+p$, shew that their L.C.M. is $(x-2)(ax+1)(bx+1)$.

CHAPTER XIII.

FRACTIONS.

96. The fraction $\frac{a}{b}$ denotes that the unit is divided into b equal parts, and that a parts of these are taken; the fraction $\frac{a-x}{a+x}$ denotes that the unit is divided into $(a+x)$ equal parts, and that $(a-x)$ of these are taken.

In the fraction $\frac{a}{b}$, a is called the *numerator* and b the *denominator.*

Every integer may be considered as a fraction with unity for its denominator; thus $p=\frac{p}{1}$.

97. The value of a fraction is not altered if the numerator and denominator be divided or multiplied by the same quantity.

$$\text{Thus}\quad \frac{a}{b}=\frac{am}{bm}=\frac{an}{bn}=\frac{\frac{a}{m}}{\frac{b}{m}}=\frac{\frac{a}{n}}{\frac{b}{n}}.$$

It follows from this that we may change the signs of both numerator and denominator of a fraction without changing its value.

$$\text{Thus}\quad \frac{a}{b}=\frac{a\times(-1)}{b\times(-1)}=\frac{-a}{-b};\quad \frac{a-x}{2a-x}=\frac{(a-x)\,(-1)}{(2a-x)(-1)}$$
$$=\frac{-a+x}{-2a+x}=\frac{x-a}{x-2a}.$$

98. A fraction is said to be in its *simplest form,* when its numerator and denominator have no common factor.

Rule for reducing a fraction to its simplest form.—"Divide both numerator and denominator by their H.C.F.; the quotients are respectively the numerator and denominator of the reduced fraction."

Ex. 1. Reduce $\dfrac{2ab(a^2-b^2)}{6(a^2b+ab^2)}$ to its lowest terms.

Here $\dfrac{2ab(a^2-b^2)}{6(a^2b+ab^2)}=\dfrac{2ab(a+b)(a-b)}{6ab(a+b)}=\dfrac{a-b}{3}$.

The H.C.F. of the numerator and denominator being $2ab(a+b)$.

Ex. 2. Reduce $\dfrac{11a^4+24a^3+125}{a^4+24a+55}$ to its lowest terms.

The H.C.F. of the numerator and denominator is a^2+4a+5. Dividing both by it the quotients are $11a^2-20a+25$ and $a^2-4a+11$.

Hence the fraction in its lowest terms $=\dfrac{11a^2-20a+25}{a^2-4a+11}$.

EXERCISE 37.

Reduce to its lowest terms :—

1. $\dfrac{x^2-a^2}{x^2-ax}$.
2. $\dfrac{a^2-3a}{9a-a^3}$.
3. $\dfrac{4a^2-9b^2}{4a^2+6ab}$.
4. $\dfrac{3ab-12a^2}{b^2-16a^2}$.
5. $\dfrac{a^4+a^2b^2+b^4}{a^3+b^3}$.
6. $\dfrac{x^4+x^2y^2+y^4}{x^6-y^6}$.
7. $\dfrac{x^2-4x+3}{x^2-7x+12}$.
8. $\dfrac{6x^2-7x-20}{4x^3-27x+5}$.
9. $\dfrac{x^2-3x-4}{x^2-4x-5}$.
10. $\dfrac{x^3-6x^2+11x-6}{x^2-3x+2}$.
11. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-2ab+b^2}$.
12. $\dfrac{6x^3-5x^2+4}{2x^3-x^2-x+2}$.
13. $\dfrac{3x^3-16x^2+23x-6}{2x^3-11x^2+17x-6}$.
14. $\dfrac{x^3-x^2y+xy^2-y^3}{x^4+2x^2y^2+y^4}$.
15. $\dfrac{x^3-3x-2}{2x^3+3x^2-1}$.
16. $\dfrac{x^5-ax^4+a^4x-a^5}{x^5-a^5}$.
17. $\dfrac{x^3-39x+70}{x^2-3x-70}$.
18. $\dfrac{x^3-19x^2+119x-245}{3x^2-38x+119}$.

19. $\dfrac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$.

20. $\dfrac{2x^3+8x^2y+16xy^2+16y^3}{8x^2+4xy-24y^2}$.

21. $\dfrac{3a^2x^4-2ax^2-1}{4a^3x^6-2a^2x^4-3ax^2+1}$.

22. $\dfrac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$.

23. $\dfrac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{a^2+b^2+2ab-c^2}$.

24. $\dfrac{a^3+b^3+c^3-3abc}{(a-b)^2+(b-c)^2+(c-a)^2}$.

25. $\dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

26. $\dfrac{a^2+ab+ac+bc}{a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc}$.

27. $\dfrac{(2a+b)^3-(a-b)^3-(a+2b)^3}{(2a+b)(a+2b)}$.

28. $\dfrac{(a+b+c)^3-a^3-b^3-c^3}{(a+b+c)(ab+ac+bc)-abc}$.

29. $\dfrac{(ab-1)^2+(a+b-2)(a+b-2ab)}{(ab+1)^2-(a+b)^2}$.

30. $\dfrac{ab+2a^2-3b^2+4bc+ac-c^2}{2a^2-9ac-5ab+4c^2-8bc-12b^2}$.

99. To reduce fractions having different denominators to equivalent fractions having the same denominator.

Rule.—"Take as common denominator the L.C.M. of all denominators; divide this L.C.M. by each denominator in turn, and multiply the quotients by the corresponding numerators; the products so obtained are the numerators of the equivalent fractions."

Note.– Compare the corresponding rule in Arithmetic.

Ex. 1. Reduce $\dfrac{1}{2x}$, $\dfrac{3}{4xy}$ and $\dfrac{5}{6y^2}$ to equivalent fractions having a common denominator.

The L.C.M. of the denominators is $12xy^2$.

Dividing this by each denominator in turn, the quotients are $6y^2$, $3y$ and $2x$; multiplying these by the corresponding numerators, we get $6y^2$, $9y$ and $10x$. Therefore the fractions are $\frac{6y^2}{12xy^2}$, $\frac{9y}{12xy^2}$ and $\frac{10x}{12xy^2}$.

Ex. 2. Reduce $\frac{2}{b^2-a^2}$, $\frac{3a}{(a+b)^2}$ and $\frac{4b^2}{(a-b)^2}$ to equivalent fractions having a common denominator.

The first fraction may be written $\frac{-2}{a^2-b^2}$ by changing the signs of both numerator and denominator.

The L.C.M. of the denominators is $(a+b)^2\times(a-b)^2$; dividing this by the corresponding denominators, we get a^2-b^2, $(a-b)^2$ and $(a+b)^2$.

Therefore the required fractions are :—

$$\frac{-2(a^2-b^2)}{(a+b)^2(a-b)^2},\quad \frac{3a(a-b)^2}{(a+b)^2(a-b)^2} \text{ and } \frac{4b^2(a+b)^2}{(a+b)^2(a-b)^2}.$$

EXERCISE 37·A.

Reduce to equivalent fractions having a common denominator :—

1. $\frac{a}{3b}$, $\frac{x}{2y}$ and $\frac{z}{6b^2y^2}$.
2. $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$.
3. $\frac{a}{bc}$, $\frac{b}{ac}$ and $\frac{c}{ab}$.
4. $\frac{1}{ab}$, $\frac{1}{bc}$ and $\frac{1}{ac}$.
5. $\frac{1}{a-b}$, $\frac{1}{b-c}$ and $\frac{1}{c-a}$.
6. $\frac{1}{(a-b)(b-c)}$, $\frac{1}{(b-c)(c-a)}$ and $\frac{1}{(a-b)(a-c)}$.
7. $\frac{1}{ab+b^2}$, $\frac{1}{a^2-ab}$ and $\frac{1}{a^2-b^2}$.
8. $\frac{1}{a^2+ab+b^2}$, $\frac{1}{a^2-ab+b^2}$ and $\frac{1}{a^6-b^6}$.

9. $\frac{1}{x^2-3x+2}$, $\frac{1}{x^2-5x+6}$ and $\frac{1}{x^2-4x+3}$.

10. $\frac{x+1}{x-1}$, $\frac{x^2+1}{x^2+x+1}$ and $\frac{x^2}{x^3-1}$.

11. $\frac{a}{a^2+ab+ac+bc}$, $\frac{b}{b^2+ab+bc+ac}$ and $\frac{c}{c^2+ab+ac+bc}$.

12. $\frac{a-b}{x^2-(a+b)x+ab}$, $\frac{b-c}{x^2-(b+c)x+bc}$

and $\frac{c-a}{x^2-(c+a)x+ca}$.

100. Addition and Subtraction of Fractions.

Rule.—"Transform the fractions into equivalent one having a common denominator; then add or subtract th numerators, as the case may be, retaining the common denomi nator."

Ex. 1. Find the sum of $\frac{1}{a}$ and $\frac{1}{b}$.

The common denominator is ab.

$\because \frac{1}{a}=\frac{b}{ab}$ and $\frac{1}{b}=\frac{a}{ab}$. $\therefore$ the sum $=\frac{b}{ab}+\frac{a}{ab}=\frac{b+a}{ab}$.

Ex. 2. Find the sum of $\frac{a}{a-b}$ and $\frac{a^2}{a^2-b^2}$.

The common denominator is a^2-b^2.

$$\therefore \frac{a}{a-b}+\frac{a^2}{a^2-b^2}=\frac{a(a+b)}{a^2-b^2}+\frac{a^2}{a^2-b^2}=\frac{a^2+ab+a^2}{a^2-b^2}.$$

$$=\frac{2a^2+ab}{a^2-b^2}$$

Ex. 3. Simplify $\frac{x+a}{x-a}-\frac{x^2-a^2}{x^2+a^2}+\frac{x^2+a^2}{x^2-a^2}$.

Here the L.C.M. of the denominators is $(x^2+a^2)(x^2-a^2)$ $=x^4-a^4$; therefore the expression

$$=\frac{(x+a)(x^2+a^2)(x+a)-(x^2-a^2)(x^2-a^2)+(x^2+a^2)(x^2+a^2)}{x^4-a^4}$$

$$=\{(x^4+a^4+2x^2a^2+2ax^3+2a^3x)-(x^4+a^4-2x^2a^2)$$
$$+(x^4+a^4+2x^2a^2)\}\div(x^4-a^4)$$
$$=\frac{x^4+2ax^3+2a^3x+6x^2a^2+a^4}{x^4-a^4}.$$

Ex. 4. Simplify $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}$.

Here $+\dfrac{b+c}{(b-a)(x-b)}=-\dfrac{b+c}{(a-b)(x-b)}$.

Therefore the L.C.M. of the denominators is $(a-b)(x-a)(x-b)$.

therefore the expression $=\dfrac{(a+c)(x-b)-(b+c)(x-a)}{(a-b)(x-a)(x-b)}$

$$=\frac{ax-ab+cx-bc-(bx-ab+cx-ac)}{(a-b)(x-a)(x-b)}$$

$$=\frac{ax-bx-bc+ac}{(a-b)(x-a)(x-b)}=\frac{x(a-b)+c(a-b)}{(a-b)(x-a)(x-b)}$$

$$=\frac{(a-b)(x+c)}{(a-b)(x-a)(x-b)}=\frac{x+c}{(x-a)(x-b)}.$$

EXERCISE 38

Simplify —

1. $\dfrac{1+x}{1+x+x^2}+\dfrac{1-x}{1-x+x^2}$.

2. $\dfrac{2a+3b}{2a-3b}+\dfrac{2a-3b}{2a+3b}$.

3. $\dfrac{2}{2x+1}-\dfrac{8}{4x+3}+\dfrac{3}{3x+2}$.

4. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a-b}{a+b}$.

5. $\dfrac{a-x}{x}+\dfrac{2a}{a-x}+\dfrac{a^3+a^2x}{x^3-a^2x}$.

6. $\dfrac{1}{x}-\dfrac{2}{x+1}+\dfrac{1}{x+2}$.

7. $\dfrac{x^2}{x^2-1}-\dfrac{x}{1-x}+\dfrac{x}{1+x}$.

8. $\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2+x^2}$.

9. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}$.

10. $\dfrac{2}{x}+\dfrac{3}{1-2x}-\dfrac{2x-3}{4x^2-1}$.

11. $\dfrac{a}{(a-b)^3}+\dfrac{ab}{(a-b)^2}-\dfrac{a}{a-b}$.

12. $\dfrac{1}{(a-2)(a-3)}+\dfrac{2}{(a-1)(3-a)}+\dfrac{1}{(a-1)(a-2)}$.

13. $\dfrac{(a^2+b^2)^2}{ab(a-b)^2}-\dfrac{a}{b}-\dfrac{b}{a}-2.$ 14. $\dfrac{x^3}{a^3+x^3}+\dfrac{c}{a+x}+\dfrac{a}{b}.$

15. $\dfrac{x+y}{y}-\dfrac{x}{x+y}-\dfrac{x^3-x^2y}{x^2y-y^3}.$ 16. $\dfrac{1}{2x+y}+\dfrac{1}{2x-y}-\dfrac{3x}{4x^2-y^2}.$

17. $\dfrac{1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)(x+3)}.$

18. $\dfrac{1}{1-x}-\dfrac{1}{(1-x)^2}+\dfrac{1}{(1-x)^3}-\dfrac{1}{(1-x)^4}.$

19. $\dfrac{x}{1-x}+\dfrac{x^2}{x^2-1}+\dfrac{x^2}{1+x^3}.$

20. $\dfrac{4}{9(a-2)}+\dfrac{5}{9(a+1)}-\dfrac{1}{3(a+1)^2}.$

21. $\dfrac{1}{x^2-2}-\dfrac{2}{x^2-1}+\dfrac{2}{x^2+1}-\dfrac{1}{x^2+2}.$

22. $\dfrac{x-1}{(x+2)(x+5)}-\dfrac{2(x+2)}{(x+5)(x-1)}+\dfrac{x+5}{(x-1)(x-2)}.$

23. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}.$

24. $\dfrac{a-b}{a+b}+\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}.$

25. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2-5x+6}+\dfrac{1}{x^2-8x+15}.$

26. $\dfrac{1-x+x^2}{1+x+x^2}-\dfrac{1+x+x^2}{1-x+x^2}+\dfrac{4x(1+x^2)}{1+x^2+x^4}.$

27. $\dfrac{(a-b)^2}{(b-c)(c-a)}+\dfrac{(b-c)^2}{(a-b)(c-a)}+\dfrac{(c-a)^2}{(a-b)(b-c)}.$

28. $\dfrac{1}{(x-a)^2}+\dfrac{2a}{(x-a)^3}+\dfrac{a^2}{(x-a)^4}.$

29. $\dfrac{3+2a}{2-a}-\dfrac{2-3a}{2+a}-\dfrac{16a-a^2}{a^2-4}.$

30. $\dfrac{1}{1+a+a^2+a^3}+\dfrac{1}{1-a-a^2+a^3}-\dfrac{2a^3}{1-a^2-a^4+a^6}.$

101. Multiplication and Division of Fractions.

I. To multiply two or more fractions :—

Rule.—" Multiply the numerators of the fractions together for a new numerator and the denominators together for a new denominator and reduce the resulting fraction to its lowest terms."

Ex. 1. Multiply $\dfrac{a^2-b^2}{4}$ by $\dfrac{a+b}{(a-b)^2}$.

$$\frac{a^2-b^2}{4}\times\frac{a+b}{(a-b)^2}=\frac{(a^2-b^2)(a+b)}{4(a-b)^2}=\frac{(a+b)(a+b)}{4(a-b)}=\frac{(a+b)^2}{4(a-b)}.$$

Ex. 2. Multiply $\dfrac{a^3-b^3}{a^3+b^3}$ by $\dfrac{(a+b)^2}{(a-b)^2}$.

$$\text{Product }=\frac{(a^3-b^3)(a+b)^2}{(a^3+b^3)(a-b)^2}=\frac{(a-b)(a^2+ab+b^2)(a+b)^2}{(a+b)(a^2-ab+b^2)\,(a-b)^2}$$

$$=\frac{(a^2+ab+b^2)(a+b)}{(a^2-ab+b^2)(a-b)}=\frac{a^3+2a^2b+2ab^2+b^3}{a^3-2a^2b+2ab^2-b^3}.$$

II. To divide one fraction by another :—

Rule.—" Invert the Divisor and proceed as in Multiplication."

Ex. 1. Divide $\dfrac{a^2-b^2}{a+2b}$ by $\dfrac{a-b}{3a+6b}$.

$$\text{Quotient }=\frac{a^2-b^2}{a+2b}\times\frac{3a+6b}{a-b}=\frac{(a+b)(a-b)3(a+2b)}{(a+2b)(a-b)}$$

$$=3(a+b).$$

Ex. 2. Divide $\dfrac{a^3+b^3}{a^3-b^3}$ by $\dfrac{a^2-ab+b^2}{a-b}$.

$$\text{Quotient }=\frac{a^3+b^3}{a^3-b^3}\times\frac{a-b}{a^2-ab+b^2}$$

$$=\frac{(a+b)(a^2-ab+b^2)(a-b)}{(a-b)(a^2+ab+b^2)(a^2-ab+b^2)}=\frac{a+b}{a^2+ab+b^2}.$$

Note. $\dfrac{\frac{a}{b}}{c}=\dfrac{a}{b}\div c=\dfrac{a}{b}\times\dfrac{1}{c}=\dfrac{a}{bc}$

and $\dfrac{a}{\frac{b}{c}}=a\div\dfrac{b}{c}=\dfrac{a}{1}\times\dfrac{c}{b}=\dfrac{ac}{b}.$

The expression $x^2+1-\dfrac{2}{x+1}$ is a *mixed* fraction;

and $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$ is a *complex* fraction. The latter is simplified thus:

$$\text{The fraction}=\frac{a}{b+\dfrac{c}{\dfrac{df+e}{f}}}=\frac{a}{b+\dfrac{cf}{df+e}}$$

$$=\frac{a}{\dfrac{bdf+be+cf}{df+e}}=\frac{a(df+e)}{bdf+be+cf}=\frac{adf+ae}{bdf+be+cf}$$

111. *To find the H.C.F. and L.C.M. of two or more fractions.*—

Rule.—"The H.C.F. of two or more fractions in their lowest terms $=\dfrac{\text{the H.C.F. of numerators}}{\text{the L.C.M. of denominators}}$; and the L.C.M. of two or more fractions in their lowest terms $=\dfrac{\text{the L.C.M. of numerators}}{\text{the H.C.F. of denominators}}$

Ex. Find the H.C.F. and L.C.M. of $\dfrac{a^2}{bc}$, $\dfrac{b^2}{ac}$, $\dfrac{c^2}{ab}$. Here the fractions are in their lowest terms.

$$\text{Their H.C.F.}=\frac{\text{the H.C.F. of } a^2,\ b^2 \text{ and } c^2}{\text{the L.C.M. of } bc,\ ac \text{ and } ab}=\frac{1}{abc}.$$

$$\text{Their L.C.M.}=\frac{\text{the L.C.M. of } a^2,\ b^2 \text{ and } c^2}{\text{the H.C.F. of } bc,\ ac \text{ and } ab}=\frac{a^2b^2c^2}{1}$$

$$=a^2b^2c^2.$$

EXERCISE 39.

Simplify:—

1. $\dfrac{a^4-b^4}{(a+b)^2}\div\dfrac{a-b}{ab+b^2}$.

2. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\div\dfrac{a^2+b^2}{a-b}$.

3. $\dfrac{x^2-9x+20}{x^2-6x}\times\dfrac{x^2-13x+42}{x^2-5x}$.

4. $\left(1-\dfrac{a-b}{a+b}\right)\times\left(2+\dfrac{2b}{a-b}\right)$.

5. $\dfrac{a^2-x^2}{a+b}\times\dfrac{a^2-b^2}{ax+x^2}\times\left(a+\dfrac{ax}{a-x}\right)$.

6. $\dfrac{a(a-x)}{a^2+2ax+x^2} \times \dfrac{x(a+x)}{a^2-2ax+x^2}$.

7. $\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right) \div \left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)$.

8. $\left(a+\dfrac{b-a}{1+ab}\right) \div \left(1-a\dfrac{b-a}{1+ab}\right)$.

9. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$

10. $\left(\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)$.

11. $\left\{\left(\dfrac{1}{a}+\dfrac{1}{b+c}\right) \div \left(\dfrac{1}{a}-\dfrac{1}{b+c}\right)\right\} \left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$.

12. $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}} \div \dfrac{1}{x+\dfrac{1}{y}}$. 13. $\left(\dfrac{2a+b}{a+b}-1\right) \div \left(1-\dfrac{a}{a+b}\right)$.

14. $\left(1-\dfrac{2ab}{a^2+b^2}\right) \div \left(\dfrac{a^3-b^3}{a-b}-3ab\right)$.

15. $\left\{a^2-\dfrac{(a^2+b^2-c^2)^2}{4b^2}\right\} \div \left\{\left(\dfrac{a+c}{b}\right)^2-1\right\}$.

16. $\left\{1-\dfrac{(a^2+b^2-c^2-d^2)^2}{4(ab+cd)^2}\right\} \div \left\{(a+b)^2-(c-d)^2\right\}$.

17. $\left\{\dfrac{a^2+x^2}{a^2-x^2}-\dfrac{a^3+x^3}{a^3-x^3}\right\} \div \left\{\dfrac{a^3-x^3}{a^3+x^3}-\dfrac{a-x}{a+x}\right\}$.

18. $\left(\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}\right)\left(\dfrac{a^2}{b^2}-1+\dfrac{b^2}{a^2}\right) \div \left(\dfrac{a^4}{b^4}+1+\dfrac{b^4}{a^4}\right)$.

19. $\left(\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}\right) \div \left\{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}\right\}$.

20. $\dfrac{2(a-b)}{2(a-b)+\dfrac{2c^2}{c+\dfrac{c^2b}{b^2-a^2}}}$.

Find the H.C.F. and L.C.M. of :—

21. $\dfrac{a^2-b^2}{a+b}$, $\dfrac{a^3-b^3}{a-b}$ and $\dfrac{a^4-b^4}{a^2+b^2}$.

22. $\dfrac{x^3+8}{(x+2)^3}$, $\dfrac{x^2-4}{(x+1)^2}$ and $\dfrac{x^3+2x^2}{x^2-4x-12}$.

23. $\dfrac{a^4-1}{(a^2+1)^2}$, $\dfrac{a^2-2a+1}{a^3+1}$ and $\dfrac{a^4-a}{a^3+2a^2+a}$.

24. $\dfrac{a^3(b-c)}{bc(a+b)(b+c)}$, $\dfrac{b^3(c-a)}{ac(b+c)(c+a)}$ and $\dfrac{c^3(a-b)}{ab(a+b)(c+a)}$.

102. The following results are important, and the student is required to work out and verify each.

1. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}=0.$

2. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}=0.$

3. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}=1.$

4. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ac}{(b-a)(b-c)}+\dfrac{ab}{(c-a)(c-b)}=1.$

5. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}=a+b+c.$

6. $\dfrac{a^4}{(a-b)(a-c)}+\dfrac{b^4}{(b-a)(b-c)}+\dfrac{c^4}{(c-a)(c-b)}=a^2+b^2+c^2$
$+ab+ac+bc.$

Ex. 1. Simplify :—

$$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-a)(b-c)}+\frac{a+b}{(c-a)(c-b)}.$$

$$\text{Expression}=\frac{a+b+c-a}{(a-b)(a-c)}+\frac{a+b+c-b}{(b-a)(b-c)}+\frac{a+b+c-c}{(c-a)(c-b)}$$

$$=(a+b+c)\left\{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}\right\}$$

$$-\left\{\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}\right\}$$

$=(a+b+c)(0)-(0)=0$...(using 1 and 2).

Ex. 2. Simplify :—

$$\frac{1}{(x-a)(a-b)(a-c)}+\frac{1}{(x-b)(b-a)(b-c)}+\frac{1}{(x-c)(c-a)(c-b)}.$$

$$\text{Expression}=\frac{1}{(x-a)(x-b)(x-c)}\left\{\frac{(x-b)(x-c)}{(a-b)(a-c)}\right.$$
$$\left.+\frac{(x-a)(x-c)}{(b-a)(b-c)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}\right\}$$

$$=\frac{1}{(x-a)(x-b)(x-c)}\left[x^2\left\{\frac{1}{(a-b)(a-c)}+\text{two similar terms}\right\}\right.$$
$$-x\left\{\frac{b+c}{(a-b)(a-c)}+\text{two similar terms}\right\}+\frac{bc}{(a-b)(a-c)}$$
$$\left.+\text{two similar terms}\right]$$

$$=\frac{1}{(x-a)(x-b)(x-c)}\left[x^2\{0\}-x\{0\}+1\right]$$

$$=\frac{1}{(x-a)(x-b)(x-c)}\text{(using 1, 4 and Example 1).}$$

Ex. 3. Simplify :—

$$\frac{(a+1)^3}{(a-b)(a-c)}+\frac{(b+1)^3}{(b-a)(b-c)}+\frac{(c+1)^3}{(c-a)(c-b)}.$$

$$\text{Expression}=\left\{\frac{a^3}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$+3\left\{\frac{a^2}{(a-b)(a-c)}+\text{two similar terms}\right\}+3\left\{\frac{a}{(a-b)(a-c)}\right.$$
$$\left.+\text{two similar terms}\right\}+\left\{\frac{1}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$=(a+b+c)+3\times1+3\times0+0=a+b+c+3\text{ (using 1, 2, 3, 5).}$$

EXERCISE 40.

Simplify :—

1. $\dfrac{(a+1)^2}{(a-b)(a-c)}+\dfrac{(b+1)^2}{(b-a)(b-c)}+\dfrac{(c+1)^2}{(c-a)(c-b)}.$

2. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2-ac}{(b-a)(b-c)}+\dfrac{c^2-ab}{(c-a)(c-b)}.$

3. $\dfrac{a^2+1}{(a-b)(a-c)}+\dfrac{b^2+1}{(b-a)(b-c)}+\dfrac{c^2+1}{(c-a)(c-b)}.$

4. $\dfrac{na^2+mbc}{(a-b)(a-c)}+\dfrac{nb^2+mac}{(b-a)(b-c)}+\dfrac{nc^2+mba}{(c-a)(c-b)}.$

5. $\dfrac{b^2+c^2}{(a-b)(a-c)}+\dfrac{c^2+a^2}{(b-a)(b-c)}+\dfrac{a^2+b^2}{(c-a)(c-b)}.$

6. $\dfrac{a(b+c)}{(a-b)(a-c)}+\dfrac{b(a+c)}{(b-a)(b-c)}+\dfrac{c(a+b)}{(c-a)(c-b)}.$

7. $\dfrac{(a+2)(a+3)}{(a-b)(a-c)}+\dfrac{(b+2)(b+3)}{(b-a)(b-c)}+\dfrac{(c+2)(c+3)}{(c-a)(c-b)}.$

8. $\dfrac{a^2+a+1}{(a-b)(a-c)}+\dfrac{b^2+b+1}{(b-c)(b-a)}+\dfrac{c^2+c+1}{(c-a)(c-b)}.$

9. $\dfrac{a^3+1}{(a-b)(a-c)}+\dfrac{b^3+1}{(b-a)(b-c)}+\dfrac{c^3+1}{(c-a)(c-b)}.$

10. $\dfrac{a^4\pm 1}{(a-b)(a-c)}+\dfrac{b^4\pm 1}{(b-a)(b-c)}+\dfrac{c^4\pm 1}{(c-a)(c-b)}.$

11. $\dfrac{1}{(a-1)(a-b)(a-c)}+\dfrac{1}{(b-1)(b-a)(b-c)}$
$\qquad\qquad+\dfrac{1}{(c-1)(c-a)(c-b)}.$

12. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}.$

13. $\dfrac{1}{(a+x)(a-b)(a-c)}+\dfrac{1}{(b+x)(b-a)(b-c)}$
$\qquad\qquad+\dfrac{1}{(c+x)(c-a)(c-b)}.$

14. $\dfrac{a}{(a-1)(a-b)(a-c)}+\dfrac{b}{(b-1)(b-a)(b-c)}$
$\qquad\qquad+\dfrac{c}{(c-1)(c-a)(c-b)}.$

15. $\dfrac{a^2}{(a-1)(a-b)(a-c)}+$ the other two similar terms.

16. $\dfrac{a}{(a-x)(a-b)(a-c)}+$ the other two similar terms.

17. $\dfrac{a^2}{(a-x)(a-b)(a-c)}+$ the other two similar terms.

18. $\dfrac{a^3}{(a-1)(a-b)(a-c)}$ + the other two similar terms.

19. $\dfrac{(a+1)^2}{(a\ \ x)(a-b)(a-c)}$ + the other two similar terms.

20. $\dfrac{ma^2+na+p}{(a+x)(a-b)(a-c)}$ + the other two similar terms.

103. Miscellaneous Examples.

The methods employed in the following examples will help the student in simplifying many difficult expressions.

1. Simplify $\dfrac{a-b}{(x-a)(x-b)}+\dfrac{b-c}{(x-b)(x-c)}+\dfrac{c-a}{(x-c)(x-a)}$.

Now $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{x-b-(x-a)}{(x-a)(x-b)}=\dfrac{1}{x-a}-\dfrac{1}{x-b}$

$$\dfrac{b-c}{(x-b)(x-c)}=\dfrac{x-c-(x-b)}{(x-b)(x-c)}=\dfrac{1}{x-b}-\dfrac{1}{x-c}$$

$$\dfrac{c-a}{(x-c)(x-a)}=\dfrac{x-a-(x-c)}{(x-c)(x-a)}=\dfrac{1}{x-c}-\dfrac{1}{x-a}$$

$\therefore$ the whole expression $=\dfrac{1}{x-a}-\dfrac{1}{x-b}+\dfrac{1}{x-b}-\dfrac{1}{x-c}$

$$+\dfrac{1}{x-c}-\dfrac{1}{x-a}=0.$$

Note Each fraction is resolved into two partial fractions

2. Simplify $\dfrac{\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}+3}{\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}}$.

The numerator $=\dfrac{a}{b+c}+1+\dfrac{b}{c+a}+1+\dfrac{c}{a+b}+1$

(splitting 3 into 3 ones)

$$=\dfrac{a+b+c}{b+c}+\dfrac{a+b+c}{c+a}+\dfrac{a+b+c}{a+b}$$

$$=(a+b+c)\left(\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right)$$

$\therefore$ the expression $=\dfrac{(a+b+c)\left(\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right)}{\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}}=a+b+c.$

3. Simplify $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(a-c)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

$$\text{Expression} = \frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}$$
$$+\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}$$
$$=\frac{a+b-c}{a+b+c}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{a+b+c}=\frac{a+b-c+b-a+c+c+a-b}{a+b+c}$$
$$=\frac{a+b+c}{a+b+c}=1.$$

4. Simplify $(a+b)\left(\dfrac{a^2+b^2-c^2}{2ab}\right)+(b+c)\left(\dfrac{b^2+c^2-a^2}{2bc}\right)$
$$+(c+a)\left(\frac{c^2+a^2-b^2}{2ac}\right).$$

$$\text{Expression} = \left(\frac{1}{2a}+\frac{1}{2b}\right)(a^2+b^2-c^2)+\left(\frac{1}{2b}+\frac{1}{2c}\right)$$
$$\times(b^2+c^2-a^2)+\left(\frac{1}{2c}+\frac{1}{2a}\right)(c^2+a^2-b^2)$$
$$=\frac{1}{2a}(a^2+b^2-c^2+c^2+a^2-b^2)+\frac{1}{2b}(a^2+b^2-c^2+b^2+c^2-a^2)$$
$$+\frac{1}{2c}(b^2+c^2-a^2+c^2+a^2-b^2)$$
$$=\frac{1}{2a}(2a^2)+\frac{1}{2b}(2b^2)+\frac{1}{2c}(2c^2)=a+b+c.$$

5. Simplify :—
$$\frac{a^3(cy-bz)^3+b^3(az-cx)^3+c^3(bx-ay)^3}{bcx^2(cy-bz)+acy^2(az-cx)+abz^2(bx-ay)}.$$

Numerator $=(acy-abz)^3+(abz-bcx)^3+(bcx-acy)^3$.

Since $acy-abz+abz-bcx+bcx-acy=0$

$\therefore\ (acy-abz)^3+(abz-bcx)^3+(bcx-acy)^3=3(acy-abz)$
$\times(abz-bcx)(bcx-acy)=3abc(cy-bz)(az-cx)(bx-ay)$.

$$\text{Denominator}=a^2b^2c^2x^2\left(\frac{cy-bz}{a^2bc}\right)+a^2b^2c^2y^2\left(\frac{az-cx}{b^2ac}\right)$$
$$+a^2b^2c^2z^2\left(\frac{bx-ay}{c^2ab}\right)=a^2b^2c^2\left\{\frac{x^2}{a^2}\left(\frac{y}{b}-\frac{z}{c}\right)+\frac{y^2}{b^2}\left(\frac{z}{c}-\frac{x}{a}\right)\right.$$
$$\left.+\frac{z^2}{c^2}\left(\frac{x}{a}-\frac{y}{b}\right)\right\}$$

$$= a^2b^2c^2\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{y}{b}-\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)$$

$[\because p^2(q-r)+q^2(r-p)+r^2(p-q)=(p-q)(q-r)(p-r)]$

$=(bx-ay)(cy-bz)(cx-az)$

$\therefore$ the given expression $= \dfrac{3abc(cy-bz)\,(az-cx)\,(bx-ay)}{(bx-ay)\,(cy-bz)\,(cx-az)}$

$=-3abc.$

6. Simplify :—

$$\frac{1}{(a+1)(a+2)}+\frac{1}{(a+2)(a+3)}+\frac{1}{(a+3)(a+4)}+\&c.$$

Taking the first two, we have $\dfrac{1}{(a+1)(a+2)}+\dfrac{1}{(a+2)(a+3)}$

$=\dfrac{a+3+a+1}{(a+1)(a+2)(a+3)}=\dfrac{2(a+2)}{(a+1)(a+2)(a+3)}=\dfrac{2}{(a+1)(a+3)}.$

Taking $\dfrac{2}{(a+1)(a+3)}$ and $\dfrac{1}{(a+3)(a+4)}$, we have $\dfrac{2}{(a+1)(a+3)}$

$+\dfrac{1}{(a+3)(a+4)}=\dfrac{2a+8+a+1}{(a+1)(a+3)(a+4)}=\dfrac{3(a+3)}{(a+1)(a+3)(a+4)}$

$=\dfrac{3}{(a+1)(a+4)}.$ Taking this and the fourth term, *viz.*, $\dfrac{1}{(a+4)(a+5)}$, we can shew that the sum $=\dfrac{4}{(a+1)(a+5)}.$

Similarly the sum of five terms $=\dfrac{5}{(a+1)(a+6)}$ and the sum of n terms $=\dfrac{n}{(a+1)(a+n+1)}.$

EXERCISE 41.

Simplify :—

1. $\dfrac{a-b}{(a+2)(b+2)}+\dfrac{b-c}{(b+2)(c+2)}+\dfrac{c-a}{(c+2)(a+2)}.$

2. $\dfrac{a^2-b^2}{(a^2+1)(b^2+1)}+\dfrac{b^2-c^2}{(b^2+1)(c^2+1)}+\dfrac{c^2-a^2}{(c^2+1)(a^2+1)}.$

3. $\dfrac{2x-3}{(x-1)(x-2)}+\dfrac{2x-5}{(x-2)(x-3)}+\dfrac{2x-4}{(x-3)(x-1)}.$

4. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+a)(b+c)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

5. $\dfrac{a-c}{(a-b)(b-c)}-\dfrac{b-a}{(b-c)(c-a)}+\dfrac{c-b}{(c-a)(a-b)}$.

6. $\dfrac{4-a-b}{(2-a)(2-b)}+\dfrac{a+c-4}{(2-a)(2-c)}+\dfrac{c-b}{(2-b)(2-c)}$.

7. $$\frac{\dfrac{a^2}{x-a}+\dfrac{b^2}{x-b}+\dfrac{c^2}{x-c}+a+b+c}{\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}}$$

8. $$\frac{\dfrac{x+y}{x-y}+\dfrac{y+z}{y-z}+\dfrac{z+x}{z-x}+3}{\dfrac{x}{x-y}+\dfrac{y}{y-z}+\dfrac{z}{z-x}}.$$

9. $$\frac{\dfrac{1}{x-c}+\dfrac{1}{a-x}+\dfrac{1}{x-b}}{\dfrac{c}{x-c}-\dfrac{a}{x-a}+\dfrac{b}{x-b}+1}.$$

10. $$\frac{\dfrac{a+b+c}{a+b+c+d}+\dfrac{a+b}{a+b+c}+\dfrac{a}{a+b}-3}{\dfrac{d}{a+b+c+d}+\dfrac{c}{a+b+c}+\dfrac{b}{a+b}}.$$

11. $$\frac{\dfrac{b}{b-a}+\dfrac{c}{c-b}+\dfrac{d}{d-c}+\dfrac{a}{a-d}-}{\dfrac{a}{b-a}+\dfrac{b}{c-b}+\dfrac{c}{d-c}+\dfrac{d}{a-d}}.$$

12. $$\frac{\dfrac{a_1}{x+a_1}+\dfrac{a_2}{x+a_2}+\dfrac{a_3}{x+a_3}+\ldots+\dfrac{a_n}{x+a_n}-n}{\dfrac{1}{x+a_1}+\dfrac{1}{x+a_2}+\dfrac{1}{x+a_3}+\ldots+\dfrac{1}{x+a_n}}.$$

13. $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{x^4-(x+1)^2}$.

14. $\dfrac{x^2-(a-b)^2}{(x+b)^2-a^2}+\dfrac{a^2-(x-b)^2}{(x+a)^2-b^2}+\dfrac{b^2-(x-a)^2}{(a+b)^2-x^2}$.

15. $\dfrac{(x+2y)^2-y^2}{(x+y)^2-4y^2}+\dfrac{(x-y)^2-4y^2}{(x-2y)^2-y^2}+\dfrac{(2x+3y)^2-y^2}{(2x+y)^2-9y^2}$.

16. $\dfrac{(a+b)}{ab}(a^2+b^2-c^2)+\dfrac{b+c}{bc}(b^2+c^2-a^2)+\dfrac{c+a}{ca}(c^2+a^2-b^2)$.

17. $\dfrac{a+b}{2ab}(a^3+b^3-c^3)+\dfrac{b+c}{2bc}(b^3+c^3-a^3)+\dfrac{c+a}{2ca}(c^3+a^3-b^3).$

18. $\dfrac{a^2+b^2}{a^2b^2}(a^4+b^4-c^4)+\dfrac{b^2+c^2}{b^2c^2}(b^4+c^4-a^4)+\dfrac{c^2+a^2}{c^2a^2}(c^4+a^4-b^4).$

19. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{a^2(b-c)+b^2(c-a)+c^2(a-b)}.$

20. $\dfrac{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}{a^3(b-c)+b^3(c-a)+c^3(a-b)}.$

21. $\dfrac{(b+c-2a)^2}{(a+b-2c)(c+a-2b)}+\dfrac{(c+a-2b)^2}{(a+b-2c)(b+c-2a)}+\dfrac{(a+b-2c)^2}{(c+a-2b)(b+c-2a)}.$

22. $$\frac{a^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b^2\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}{a\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}$$

23. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{ab(a-b)+bc(b-c)+ca(c-a)}.$

24. $\dfrac{1}{(x+1)(2x+1)}+\dfrac{1}{(2x+1)(3x+1)}+\dfrac{1}{(3x+1)(4x+1)}+\dfrac{1}{(4x+1)(5x+1)}.$ What is the sum of n terms of the above?

CHAPTER XIV.

IDENTITIES AND FRACTIONS—*Continued.*

104. If two algebraic expressions are identically equal for all values of the letters x, y, &c., involved in them, then the co-efficients of the like powers of x, y, &c., in each expression are equal.

If $px+qy=ax+by$ for *all* values of x and y, then $p=a$ and $q=b$.

Since $px+qy=ax+by$ for all values of x, it must be true when $x=0$, $\therefore qy=by$, $\therefore q=b$ and $\therefore p=a$.

If $A+Bx+Cx^2+Dx^3+\&c. =a+bx+cx^2+dx^3+\&c.$, for *all* values of x, then $A=a$, $B=b$, $C=c$, $D=d$, &c.

Since it is true for *all* values of x, it must be true when $x=0$.

$\therefore \quad A=a, \quad \therefore \quad Bx+Cx^2+Dx^3+\&c.=bx+cx^2+dx^3+\&c.$ Dividing both sides by x, $B+Cx+Dx^2+\&c.=b+cx+dx^2+\&c.$ By putting $x=0$, it may be shewn that $B=b$; similarly $C=c$, and so on. This is the *Principle of Indeterminate Co-efficients.*

Note.—If $a_1+a_2x+a_3x^2+\&c.=0$ for all values of x, then $a_1=0$, $a_2=0$, $a_3=0$, &c.

105. Application of the above Principle.

Ex. 1. To resolve $\dfrac{2}{(x-1)(x-3)}$ into two partial fractions.

Let $\dfrac{2}{(x-1)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-3}$, where A and B do not contain x. Then $\dfrac{2}{(x-1)(x-3)}=\dfrac{A(x-3)+B(x-1)}{(x-1)(x-3)}$.

$\therefore 2=A(x-3)+B(x-1)$. Since this is true for all values of x, it must be true when $x=1$, $\therefore 2=A(1-3)$, $\therefore 2=-2A$.

$\therefore A=-1$. Similarly when $x=3$, $B=1$.

$$\therefore \quad \frac{2}{(x-1)(x-3)}=-\frac{1}{x-1}+\frac{1}{x-3}.$$

Ex. 2. To resolve $\dfrac{x^2}{(x-1)(x-2)(x-3)}$ into three partial fractions.

Let $\dfrac{x^2}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$, where A, B and C do not contain x. Then we have $x^2=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$.

Since this is true for all values of x, it must be true when $x=1$, $\therefore\ 1=A(1-2)(1-3)$, $\therefore\ 1=2A$, $\therefore\ A=\frac{1}{2}$.

Similarly, when $x=2$, $4=B(2-1)(2-3)$, $\therefore\ B=-4$.

And when $x=3$, $9=C(3-1)(3-2)$, $\therefore C=\frac{9}{2}$.

$$\therefore\ \frac{x^2}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}.$$

Note.—The student is referred to Chapters XVII and XXX for more examples of the Application of this Principle.

EXERCISE 42.

Resolve each of the following into two partial fractions :—

1. $\dfrac{a-b}{(x-a)(x-b)}$.
2. $\dfrac{3x}{(x-4)(x+2)}$.
3. $\dfrac{2}{(x+6)(x+2)}$.
4. $\dfrac{x^2-4x}{x^3+8}$.
5. $\dfrac{x}{(2x-1)(3x-1)}$.
6. $\dfrac{2x}{x^2-(a+b)^2}$.
7. $\dfrac{px+q}{(x+a)(x+b)}$.

Resolve each of the following into three partial fractions :—

8. $\dfrac{3}{(a+1)(a+2)(a+3)}$.
9. $\dfrac{ax^2+bx+c}{(x-p)(x-q)(x-r)}$.
10. $\dfrac{10y^2-41y+37}{(y-1)(y-2)(y-3)}$.
11. $\dfrac{2x^2-10x+14}{(x-2)(x-3)(x-4)}$.
12. $\dfrac{12x^2+73x+106}{(x+2)(x+4)(x+3)}$.

106. Examples worked out.

1. If $x=\dfrac{4ab}{a+b}$, shew that $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}=2$.

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=\frac{x-2a+4a}{x-2a}+\frac{x-2b+4b}{x-2b}$$

$$=1+\frac{4a}{x-2a}+1+\frac{4b}{x-2b}=2+4\left(\frac{a}{x-2a}+\frac{b}{x-2b}\right)$$

$$=2+4\left\{\frac{ax-2ab+bx-2ab}{(x-2a)(x-2b)}\right\}=2+4\left\{\frac{x(a+b)-4ab}{(x-2a)(x-2b)}\right\}$$

$$=2+4\left\{\frac{\frac{4ab}{a+b}(a+b)-4ab}{(x-2a)(x-2b)}\right\}=2+4\left\{\frac{4ab-4ab}{(x-2a)(x-2b)}\right\}$$

$$=2+4\left\{\frac{0}{(x-2a)(x-2b)}\right\}=2+0=2.$$

2. Find the value of $\dfrac{(x-a)(x-b)}{(x-a-b)^2}$ when $x=\dfrac{a^2+ab+b^2}{a+b}$.

$$x=\frac{a^2+ab+b^2}{a+b}=\frac{a(a+b)+b^2}{a+b}=a+\frac{b^2}{a+b}.$$

$$\therefore x-a=\frac{b^2}{a+b}\dots\dots\dots\dots(1).$$

$$x=\frac{b(a+b)+a^2}{a+b}=b+\frac{a^2}{a+b}\qquad\therefore x-b=\frac{a^2}{a+b}\dots\dots(2).$$

Again $x=\dfrac{a^2+2ab+b^2-ab}{a+b}=a+b-\dfrac{ab}{a+b}$

$$\therefore x-a-b=-\frac{ab}{a+b}\qquad\therefore (x-a-b)^2=\frac{a^2b^2}{(a+b)^2}\dots\dots\dots(3).$$

From (1), (2) and (3), $\dfrac{(x-a)(x-b)}{(x-a-b)^2}=\dfrac{a^2}{a+b}\times\dfrac{b^2}{a+b}\div\dfrac{a^2b^2}{(a+b)^2}$

$$=\frac{a^2b^2}{(a+b)^2}\times\frac{(a+b)^2}{a^2b^2}=1.$$

3. If $b=\sqrt{ac}$, shew that $\dfrac{b+3a}{c+3b}=\dfrac{a+3b}{b+3c}$.

Now $\dfrac{b+3a}{c+3b}=\dfrac{\sqrt{ac}+3a}{c+3\sqrt{ac}}=\dfrac{\sqrt{a}(\sqrt{c}+3\sqrt{a})}{\sqrt{c}(\sqrt{c}+3\sqrt{a})}=\dfrac{\sqrt{a}}{\sqrt{c}}\dots\dots(1).$

Again $\dfrac{a+3b}{b+3c}=\dfrac{a+3\sqrt{ac}}{\sqrt{ac}+3c}=\dfrac{\sqrt{a}(\sqrt{a}+3\sqrt{c})}{\sqrt{c}(\sqrt{a}+3\sqrt{c})}=\dfrac{\sqrt{a}}{\sqrt{c}}\dots\dots(2).$

Since $\dfrac{b+3a}{c+3b}$ and $\dfrac{a+3b}{b+3c}$ are each equal to $\dfrac{\sqrt{a}}{\sqrt{c}}$, therefore, $\dfrac{b+3a}{c+3b}=\dfrac{a+3b}{b+3c}$.

4. If $\dfrac{1}{c}=\dfrac{a+b}{b-ab}$, shew that $\dfrac{1}{b}=\dfrac{a+c}{b-ac}$ and $\dfrac{1}{a}=\dfrac{b+c}{b-bc}$.

Now $\dfrac{1}{c}=\dfrac{a+b}{b-ab}$ $\quad\therefore b-ab=c(a+b)$.

$\therefore b-ab=ac+bc \quad \therefore b=ab+ac+bc$

$\therefore b-ac=ab+bc=b(a+c) \quad \therefore b=\dfrac{b-ac}{a+c}$

$\therefore \dfrac{1}{b}=\dfrac{a+c}{b-ac}$.

Again since $b=ab+ac+bc$, $\quad\therefore b-bc=ab+ac=a(b+c)$

$\therefore a=\dfrac{b-bc}{b+c} \qquad \therefore \dfrac{1}{a}=\dfrac{b+c}{b-bc}$.

5. If $x=\dfrac{a-b}{1+ab}$, $y=\dfrac{b-c}{1+bc}$ and $z=\dfrac{c-a}{1+ca}$, shew that $x+y+z=xyz$.

$$\text{Now } x+y+z=\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}$$

$$=\frac{a-b}{1+ab}+\frac{b-a+a-c}{1+bc}+\frac{c-a}{1+ca}$$

$$=(a-b)\left\{\frac{1}{1+ab}-\frac{1}{1+bc}\right\}+(c-a)\left\{\frac{1}{1+ca}-\frac{1}{1+bc}\right\}$$

$$=(a-b)\frac{b(c-a)}{(1+ab)(1+bc)}+(c-a)\frac{c(b-a)}{(1+ca)(1+bc)}$$

$$=\frac{(a-b)(c-a)}{(1+bc)}\left\{\frac{b}{1+ab}-\frac{c}{1+ca}\right\}$$

$$=\frac{(a-b)(c-a)}{1+bc}\left\{\frac{b-c}{(1+ab)(1+ca)}\right\}$$

$$=\frac{(a-b)(b-c)(c-a)}{(1+ab)(1+bc)(1+ca)}=xyz.$$

EXERCISE 43

Find the value of :—

1. $\dfrac{x(x-a)+x(x-b)}{(x-a)(x-b)}$ when $x=\dfrac{2ab}{a+b}$.

2. $\dfrac{x-a}{x-b}-\left(\dfrac{x-2a}{x-2b}\right)^2$ when $x=\dfrac{ab}{a+b}$.

3. $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

4. $\dfrac{a(a-c)}{(a-x)(a+c)}$ when $x=\dfrac{2ac}{a+c}$.

5. $\left(\dfrac{x-2a}{x-2b}\right)^3-\dfrac{x-4a+2b}{x+2a-4b}$ when $x=a+b$.

6. $\dfrac{(a-x)(b-x)}{(c+x)(d+x)}$ when $x=\dfrac{ab-cd}{a+b+c+d}$.

7. $\dfrac{1}{x-a}+\dfrac{1}{x-b}-\dfrac{1}{b}$ when $x=\dfrac{2ab}{a+b}$.

8. $\dfrac{(x-a)(x-b)(x+2a+2b)}{(x+2a)(x+2b)(x-a-b)}$ when $x=2(a+b)$.

9. $\dfrac{(x-a)(x-b)(a+b)}{(x-a-b)(b-a)}$ when $x=\dfrac{a^2+ab+b^2}{a+b}$.

10. $\dfrac{x+a}{x+b}\left\{\dfrac{2x+b+c}{2x+a+c}\right\}^2$ when $x=\dfrac{c^2-ab}{a+b-2c}$.

11. $\dfrac{bx-a}{ax-b}$ when $x=\dfrac{a^2+b^2}{2ab}$.

12. $\left(\dfrac{x+2c}{x+2d}\right)\left(\dfrac{x-c}{x-d}\right)^2$ when $x=\dfrac{2(c^2+cd+d^2)}{3(c+d)}$.

13. $\dfrac{x^2-y^2+x}{y^2-x^2+y}$ when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$.

14. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}$ when $x=\dfrac{ab}{a+b}$.

15. $\dfrac{a(x-c)}{(x-a)(a-b)}-\dfrac{b(x-c)}{(x-b)(a-b)}$ when $x=\dfrac{ab}{a+b-c}$.

16. $\dfrac{1}{x-a}+\dfrac{1}{x-b}-\dfrac{a-b}{x^2-ab}$ when $x=\dfrac{1}{\dfrac{1}{2a}+\dfrac{1}{2b}}$.

17. $\dfrac{(px-a-b)(qx-b-d)}{(px-a-c)(qx-c-d)}$ when $x=\dfrac{a+b+c+d}{p+q}$.

18. $\dfrac{(a+b)(x-a)(b-x)}{ab(2x-a-b)}$ when $x=\dfrac{ab}{a+b}$.

19. $\dfrac{x^2+a^2-b^2}{2ax}+\dfrac{a^2+b^2-x^2}{2ab}+\dfrac{b^2+x^2-a^2}{2bx}$ when $x=a+b$.

20. $\dfrac{x+y-1}{x-y+1}$ when $x=\dfrac{a+1}{ab+1}$ and $y=\dfrac{ab+a}{ab+1}$.

Shew that—

21. $\dfrac{2bc+(b^2+c^2-a^2)}{2bc-(b^2+c^2-a^2)}=\dfrac{s(s-a)}{(s-b)(s-c)}$ if $2s=a+b+c$.

22. $\dfrac{ab}{(x-a)(x-b)}+\dfrac{bc}{(x-b)(x-c)}+\dfrac{ca}{(x-c)(x-a)}=0$ if $\dfrac{1}{x}$ $=\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

23. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if $\dfrac{x}{a}=\dfrac{p^2-\dfrac{1}{p^2}}{p^2+\dfrac{1}{p^2}}$ and $\dfrac{b}{y}=\dfrac{1}{2}\left(p^2+\dfrac{1}{p^2}\right)$.

24. $\dfrac{1}{m(n-x)}+\dfrac{1}{n(a-x)}+\dfrac{1}{m(x-a)}=0$ if x $=\dfrac{n}{m}(m-n+a)$.

25. $\dfrac{(x-a)(x-b)}{(y-a)(y-b)}=\dfrac{x}{y}$ if $x=\dfrac{a+b}{2}$ and $y=\dfrac{2ab}{a+b}$.

107. Examples worked out.

1. If $a+b+c=0$, shew that—

i. $\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ac}=0$; (ii) $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^2=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$;

(iii) $\dfrac{1}{a^2+b^2-c^2}+\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{c^2+a^2-b^2}=0$;

(iv) $\dfrac{a}{2a^2+bc}+\dfrac{b}{2b^2+ac}+\dfrac{c}{2c^2+ab}=0.$

(i) $\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ac}=\dfrac{c+a+b}{abc}=\dfrac{0}{abc}=0$; $\because a+b+c=0$;

(ii) $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^2=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}+2\left(\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}\right)$

$=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$. $\because \dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}=0$ [by (i)].

From this we may deduce $\left(\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}\right)^2$

$=\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$.

(iii) Since $a+b+c=0$ $\therefore$ $a+b=-c$ $\therefore$ $(a+b)^2=c^2$.
$\therefore$ $a^2+b^2+2ab=c^2$ $\therefore$ $a^2+b^2-c^2=-2ab$.
Similarly $b^2+c^2-a^2=-2bc$ and $c^2+a^2-b^2=-2ca$.

Hence $\dfrac{1}{a^2+b^2-c^2}+\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{c^2+a^2-b^2}=-\dfrac{1}{2ab}$
$-\dfrac{1}{2bc}-\dfrac{1}{2ca}=-\dfrac{1}{2}\left(\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ac}\right)=0$ $\because$ $\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}=0$ [by (i)].

(iv) Since $a+b+c=0$, $\therefore$ $a=-(b+c)$. $\therefore$ $a^2=-a(b+c)$
$\therefore a^2+a^2+bc=a^2+bc-a(b+c)$. $\therefore 2a^2+bc=a^2+bc$
$-ab-ac=a(a-b)-c(a-b)=(a-b)(a-c)$.
Similarly $2b^2+ac=(b-a)(b-c)$ and $2c^2+ab=(c-a)$
$\times(c-b)$.

Hence $\dfrac{a}{2a^2+bc}+\dfrac{b}{2b^2+ac}+\dfrac{c}{2c^2+ab}=\dfrac{a}{(a-b)(a-c)}$
$+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}=0$ [by (2) of Art. 102].

2. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, then shew that

(i) $a=-b$, or $b=-c$, or $c=-a$;

(ii) $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}$; (iii) $\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1}{a^5+b^5+c^5}$;

and generally (iv) $\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$.

(i) Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$

$\therefore \frac{ab+bc+ac}{abc}=\frac{1}{a+b+c}$ $\therefore (a+b+c)(ab+ac+bc)=abc$

$\therefore (a+b+c)(ab+ac+bc)-abc=0$ $\therefore (a+b)(b+c)(c+a)=0$

$\therefore a+b=0$ $\therefore a=-b$; or $b+c=0$, $\therefore b=-c$; or $c+a=0$ $\therefore c=-a$.

(ii) We have shewn that $a=-b$ $\therefore a^3=-b^3$

$\therefore \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=-\frac{1}{b^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}$

$\because a^3+b^3=0$

(iii) Since $a=-b$ $\therefore a^5=-b^5$

$\therefore \frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=-\frac{1}{b^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1}{a^5+b^5+c^5}$ $\because a^5+b^5=0$.

(iv) Since $a=-b$, $\therefore a^{2n+1}=-b^{2n+1}$ $\because 2n+1$ is odd.

$\therefore \frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=-\frac{1}{b^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{c^{2n+1}}$

$=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$ $\because a^{2n+1}+b^{2n+1}=0$.

3. If $ab+ac+bc=1$, shew that—

(i) $\frac{a^2+1}{(a+b)(a+c)}+\frac{b^2+1}{(b+c)(b+a)}+\frac{c^2+1}{(c+a)(c+b)}$

$=3(ab+ac+bc)$.

(ii) $\frac{a+b}{1-ab}+\frac{b+c}{1-bc}+\frac{c+a}{1-ca}=\frac{(a+b)(b+c)(c+a)}{(1-ab)(1-bc)(1-ca)}$.

(i) Since $1=ab+ac+bc$, $\therefore 1+a^2=a^2+ab+ac+ba$
$=a(a+b)+c(a+b)=(a+b)(a+c)$.

Similarly $1+b^2=(b+a)(b+c)$; and $1+c^2=(c+a)(c+b)$.

Hence the given expression $=\dfrac{(a+b)(a+c)}{(a+b)(a+c)}+\dfrac{(b+a)(b+c)}{(b+a)(b+c)}$

$+\dfrac{(c+a)(c+b)}{(c+a)(c+b)}=1+1+1=3=3(ab+bc+ac)$.

(ii) Since $ab+ac+bc=1$, $\therefore 1-ab=ac+bc=c(a+b)$.

$\therefore \dfrac{a+b}{1-ab}=\dfrac{1}{c}$. Similarly $1-bc=ab+ac=a(b+c)$

$\therefore \dfrac{b+c}{1-bc}=\dfrac{1}{a}$ and $\dfrac{c+a}{1-ca}=\dfrac{1}{b}$

$\therefore \dfrac{a+b}{1-ab}+\dfrac{b+c}{1-bc}+\dfrac{c+a}{1-ca}=\dfrac{1}{c}+\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{ab+ac+bc}{abc}=\dfrac{1}{abc}$

$$=\frac{1}{c}\cdot\frac{1}{a}\cdot\frac{1}{b}=\frac{a+b}{1-ab}\cdot\frac{b+c}{1-bc}\cdot\frac{c+a}{1-ca}.$$

4. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=0$, then shew that—

(i) $\dfrac{a^2}{(a+b)(a+c)}+\dfrac{b^2}{(b+c)(b+a)}+\dfrac{c^2}{(c+a)(c+b)}=3$;

(ii) $\dfrac{ab}{a+b}+\dfrac{bc}{b+c}+\dfrac{ca}{c+a}+a+b+c=0$.

(i) Since $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=0$, $\therefore \dfrac{ab+ac+bc}{abc}=0$.

$\therefore ab+ac+bc=0$ $\therefore a^2+ab+bc+ca=a^2$

$\therefore a(a+b)+c(a+b)=a^2$ $\therefore (a+b)(a+c)=a^2$. Similarly, by adding b^2 and c^2 to both sides of $ab+ac+bc=0$, we can get $(b+c)(b+a)=b^2$ and $(c+a)(c+b)=c^2$.

$\therefore \dfrac{a^2}{(a+b)(a+c)}+\dfrac{b^2}{(b+a)(b+c)}+\dfrac{c^2}{(c+a)(c+b)}=\dfrac{a^2}{a^2}+\dfrac{b^2}{b^2}+\dfrac{c^2}{c^2}$

$$=1+1+1=3.$$

(ii) $\dfrac{ab}{a+b}+\dfrac{bc}{b+c}+\dfrac{ca}{c+a}+a+b+c=\dfrac{ab}{a+b}+c+\dfrac{bc}{b+c}+a$

$+\dfrac{ca}{c+a}+b=\dfrac{ab+ac+bc}{a+b}+\dfrac{bc+ab+ac}{b+c}+\dfrac{ca+bc+ab}{c+a}=\dfrac{0}{a+b}$

$+\dfrac{0}{b+c}+\dfrac{0}{c+a}=0.$

5. If $a+b+c+d=0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0$, prove that
$$a^3+b^3+c^3+d^3=0.$$

Since $a+b+c+d=0$, $\therefore\ a+b=-(c+d)$.

Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0$, $\therefore\ \frac{1}{a}+\frac{1}{b}=-\left(\frac{1}{c}+\frac{1}{d}\right)$

$\therefore\ \frac{a+b}{ab}=-\frac{c+d}{cd}$ $\qquad\therefore\ \frac{1}{ab}=\frac{1}{cd}$ $\qquad\therefore\ ab=cd$.

Again $(a+b)^3=-(c+d)^3$ $\qquad\therefore\ a^3+b^3+3ab(a+b)$ $=-c^3-d^3-3cd(c+d)$ $\qquad\therefore\ a^3+b^3+c^3+d^3=-3ab(a+b)$ $-3cd(c+d)=-3cd(a+b)+3cd(a+b)=0$.

6. If $a+b+c=abc$, shew that $\frac{a+b}{ab-1}+\frac{b+c}{bc-1}+\frac{c+a}{ca-1}=abc$.

Since $a+b+c=abc$, $\therefore\ a+b=abc-c=c(ab-1)$.

Similarly, $b+c=a(bc-1)$ and $c+a=b(ac-1)$.

Hence $\frac{a+b}{ab-1}+\frac{b+c}{bc-1}+\frac{c+a}{ca-1}=\frac{c(ab-1)}{ab-1}+\frac{a(bc-1)}{bc-1}+\frac{b(ca-1)}{ca-1}$ $=c+a+b=abc$.

7. If $x+y=az$, $y+z=bx$ and $z+x=cy$, find the value of $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$.

Since $x+y=az$, $\therefore\ x+y+z=az+z=z(a+1)$

$\therefore\ a+1=\frac{x+y+z}{z}$ $\qquad\therefore\ \frac{1}{a+1}=\frac{z}{x+y+z}$.

Similarly $\frac{1}{b+1}=\frac{x}{x+y+z}$ and $\frac{1}{c+1}=\frac{y}{x+y+z}$

$$\therefore\ \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{z}{x+y+z}+\frac{x}{x+y+z}+\frac{y}{x+y+z}$$
$$=\frac{x+y+z}{x+y+z}=1.$$

8. If $a+b+c=1$, $ab+ac+bc=\frac{1}{3}$, $abc=\frac{1}{27}$, then $\frac{1}{a+bc}$ $+\frac{1}{b+ac}+\frac{1}{c+ab}=\frac{27}{4}$.

Since $a+b+c=1$, $\therefore a=1-(b+c)$. $\therefore a+bc=1-(b+c)+bc=(1-b)(1-c)$. Similarly $b+ac=(1-a)(1-c)$ and $c+ab=(1-a)(1-b)$.

$$\therefore \frac{1}{a+bc}+\frac{1}{b+ac}+\frac{1}{c+ab}=\frac{1}{(1-b)(1-c)}+\frac{1}{(1-a)(1-c)}+\frac{1}{(1-a)(1-b)}$$

$$=\frac{1-a+1-b+1-c}{(1-a)(1-b)(1-c)}=\frac{3-(a+b+c)}{1-(a+b+c)+(ab+ac+bc)-abc}$$

$$=\frac{3-1}{1-1+\frac{1}{3}-\frac{1}{27}}=\frac{2}{\frac{8}{27}}=\frac{27}{4}.$$

9. If $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=1$, then (i) $\frac{a^2}{b+c}+\frac{b^2}{a+c}+\frac{c^2}{a+b}=0$; (ii) $\frac{a^3}{b+c}+\frac{b^3}{a+c}+\frac{c^3}{a+b}=-(a^2+b^2+c^2)$.

(i) Now $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=1$

$$\therefore \frac{a(a+b+c)}{b+c}+\frac{b(a+b+c)}{a+c}+\frac{c(a+b+c)}{a+b}=a+b+c$$

$$\therefore a+\frac{a^2}{b+c}+b+\frac{b^2}{a+c}+c+\frac{c^2}{a+b}=a+b+c$$

$$\therefore \frac{a^2}{b+c}+\frac{b^2}{a+c}+\frac{c^2}{a+b}=(a+b+c)-(a+b+c)=0.$$

(ii) $\therefore \frac{a^2(a+b+c)}{b+c}+\frac{b^2(a+b+c)}{a+c}+\frac{c^2(a+b+c)}{a+b}=0$

$$\therefore a^2+\frac{a^3}{b+c}+b^2+\frac{b^3}{a+c}+c^2+\frac{c^3}{a+b}=0$$

$$\therefore \frac{a^3}{b+c}+\frac{b^3}{a+c}+\frac{c^3}{a+b}=-(a^2+b^2+c^2).$$

10. If $abc=1$, prove that $\frac{1}{1+a+\frac{1}{b}}+\frac{1}{1+c+\frac{1}{a}}+\frac{1}{1+b+\frac{1}{c}}=1.$

Now $1+a+\frac{1}{b}=abc+a+ac=a(bc+1+c)$.

$1+c+\frac{1}{a}=1+c+bc$ and $1+b+\frac{1}{c}=\frac{c+bc+1}{c}$

$$\therefore \frac{1}{1+a+\frac{1}{b}}+\frac{1}{1+c+\frac{1}{a}}+\frac{1}{1+b+\frac{1}{c}}=\frac{1}{a(bc+1+c)}+\frac{1}{1+c+bc}$$

$$+\frac{1}{\frac{c+bc+1}{c}}=\frac{1}{bc+1+c}\left(\frac{1}{a}+1+c\right)=\frac{1}{bc+1+c}(bc+1+c)=1.$$

EXERCISE 44.

If $a+b+c=0$, prove the following identities :—

1. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=0$; $\frac{a^2+b^2+c^2}{\sqrt{(a^2b^2+b^2c^2+c^2a^2)}}=2$.

2. $\frac{a+b}{(a+c)(b+c)}+\frac{b+c}{(a+b)(a+c)}+\frac{c+a}{(a+b)(b+c)}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

3. $\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)+\frac{a+b}{ab}(a^2+b^2-c^2)=0$.

4. $a^3(b-c)+b^3(c-a)+c^3(a-b)=0$.

5. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=0$.

6. $\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=9$.

7. $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}=1$.

8. $\frac{a^3}{2a^2+bc}+\frac{b^3}{2b^2+ac}+\frac{c^3}{2c^2+ab}=0$ and $\frac{a(b^3-c^3)}{b-c}+\frac{b(c^3-a^3)}{c-a}+\frac{c(a^3-b^3)}{a-b}=0$.

9. $\frac{a^4}{2a^2+bc}+\frac{b^4}{2b^2+ac}+\frac{c^4}{2c^2+ab}=-(ab+ac+bc)$.

10. $$\frac{(a^2-bc)^2}{(b^2-ac)(c^2-ab)}+\frac{(b^2-ac)^2}{(c^2-ab)(a^2-bc)}+\frac{(c^2-ab)^2}{(a^2-bc)(b^2-ac)}=3.$$

11. $$\frac{2a}{b^2+c^2-5a^2}+\frac{2b}{a^2+c^2-5b^2}+\frac{2c}{a^2+b^2-5c^2}=0.$$

12. $$\frac{a^4}{5bc+2(b^2+c^2)}+\frac{b^4}{5ac+2(a^2+c^2)}+\frac{c^4}{5ab+2(a^2+b^2)}=-(ab+ac+bc).$$

13. $$\frac{1}{(a^2+2bc)(b^2+2ac)}+\frac{1}{(b^2+2ac)(c^2+2ab)}+\frac{1}{(c^2+2ab)(a^2+2bc)}=0.$$

If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, prove that—

14. $$\left(\frac{1}{a+b+c}\right)^3=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}.$$

15. $$\left(\frac{1}{a+b+c}\right)^{2n+1}=\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}.$$

16. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2}{a+b+c}$, then $(a+b)(b+c)(c+a)=abc$.

17. If $ab+ac+bc=1$, then $\left(1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right)^2$

$$=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)};\quad \frac{1-bc}{b+c}+\frac{1-ab}{a+b}+\frac{1-ca}{c+a}=a+b+c.$$

If $a+b+c+d=0$, prove that—

18. $a^2+b^2-c^2-d^2=2(cd-ab)$.

19. $a^3+b^3+c^3+d^3=3(c+d)(ab-cd)=3(a+b)(cd-ab)$.

20. $a^4+b^4+c^4+d^4=2\{(ab-cd)^2+(ac-bd)^2+(ad+bc)^2\}-4abcd$.

21. $$\frac{1}{5}(a^5+b^5+c^5+d^5)=\frac{a^3+b^3+c^3+d^3}{3}\times\frac{a^2+b^2+c^2+d^2}{2}.$$

If $a+b+c=abc$, prove that—

22. $\dfrac{a}{1-a^2}+\dfrac{b}{1-b^2}+\dfrac{c(1-ab)^2}{(1-a^2)(1-b^2)}=0.$

23. $\dfrac{b}{1-b^2}+\dfrac{c}{1-c^2}+\dfrac{a(1-bc)^2}{(1-b^2)(1-c^2)}=0.$

24. $\dfrac{c}{1-c^2}+\dfrac{a}{1-a^2}+\dfrac{b(1-ca)^2}{(1-c^2)(1-a^2)}=0.$

25. $\dfrac{2a}{1-a^2}+\dfrac{2b}{1-b^2}+\dfrac{2c}{1-c^2}=\dfrac{2a}{1-a^2}\cdot\dfrac{2b}{1-b^2}\cdot\dfrac{2c}{1-c^2}.$

26. $\dfrac{3a-a^3}{1-3a^2}+\dfrac{3b-b^3}{1-3b^2}+\dfrac{3c-c^3}{1-3c^2}=\dfrac{3a-a^3}{1-3a^2}\cdot\dfrac{3b-b^3}{1-3b^2}\cdot\dfrac{3c-c^3}{1-3c^2}.$

27. $\dfrac{a+b}{1-ab}+\dfrac{b+c}{1-bc}+\dfrac{c+a}{1-ca}=\dfrac{a+b}{1-ab}\cdot\dfrac{b+c}{1-bc}\cdot\dfrac{c+a}{1-ca}.$

28. If $x+y+z=0$ and $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$, then $x^2+y^2+z^2=0$ and $x^4+y^4+z^4=0$.

29. If $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$, then $\dfrac{1}{x^2-yz}+\dfrac{1}{y^2-xz}+\dfrac{1}{z^2-xy}=0.$

30. If $x=\dfrac{a-b}{a+b}$, $y=\dfrac{b-c}{b+c}$ and $z=\dfrac{c-a}{c+a}$,

then $\dfrac{1+x}{1-x}\cdot\dfrac{1+y}{1-y}\cdot\dfrac{1+z}{1-z}=1.$

31. If $x=\dfrac{b-c}{a}$, $y=\dfrac{c-a}{b}$ and $z=\dfrac{a-b}{c}$, then $x+y+z+xyz=0$.

32. If $x+\dfrac{1}{y}=1$ and $z+\dfrac{1}{x}=1$, then $y+\dfrac{1}{z}=1$.

33. If $\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}=0$, then $(a-b)^2+(b-c)^2+(c-a)^2=0$.

34. If $ax=y$, $by=x$, then $\dfrac{1}{a+1}+\dfrac{1}{b+1}=1$.

35. If $x+y=ax$, $y+z=by$ and $z+x=cz$, then $(a-1)(b-1)\times(c-1)=1$.

36. If $x+y+z=-4$ and $xy+yz+zx=5$, then $\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=0$.

37. If $x+y=az$, $y+z=bx$ and $z+x=cy$, then $(a+1)(b+1)\times(c+1)=0$ if $x+y+z=0$.

38. If $a+b+c=1$, $ab+ac+bc=2$ and $a^3+b^3+c^3=3$, then $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{45}{64}$.

39. If $\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}+\frac{d}{1-d}=1$, shew that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}+\frac{1}{1-d}=5$.

40. If $\frac{x}{y}+\frac{y}{x}=1$, then $\frac{x^3}{y}+xy+\frac{y^3}{x}=0$.

41. If $(a+d)(b+c)=(1-ad)(1-bc)$, then $(a+c)(b+d)=(1-ac)(1-bd)$.

42. If $x+y+z=4$ and $xy+yz+zx=5$, shew that $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0$.

43. If $x+y+z=0$, then $\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=3$.

44. If $a+b+c+2=abc$, then $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1$.

45. If $a+b+c=0$, then $a^4+b^4+c^4+(ab+ac+bc)\times(a^2+b^2+c^2)=0$.

46. If $x=\frac{b}{c}-\frac{c}{a}$, $y=\frac{c}{a}-\frac{a}{b}$ and $z=\frac{a}{b}-\frac{b}{c}$, shew that

$$2(x^2y^2+y^2z^2+z^2x^2)-x^4-y^4-z^4=0.$$

47. If $x=\frac{a}{b+c}$, $y=\frac{b}{a+c}$ and $z=\frac{c}{a+b}$, then $xy+yz+xz+2xyz=1$.

48. If $x+y+z=du$, $y+z+u=ax$, $z+x+u=by$ and $x+y+u=cz$, then $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}+\dfrac{1}{1+d}=1$.

49. If $X=ax+by+cz$, $Y=bx+cy+az$ and $Z=cx+ay+bz$, shew that $(a^2+b^2+c^2-ab-ac-bc)(x^2+y^2+z^2-xy-yz-zx)$ $=X^2+Y^2+Z^2-XY-XZ-YZ$.

50. If $a+b+c=0$, then $\dfrac{a^7+b^7+c^7}{7}\times\dfrac{a^3+b^3+c^3}{3}$

$$=\left(\frac{a^5+b^5+c^5}{5}\right)^2.$$

51. If $2s=a+b+c+d$, then $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$ $=16(s-a)(s-b)(s-c)(s-d)$.

52. If $a_1+a_2+a_3+\ldots\ldots a_n=\dfrac{n}{2}s$, then $(s-a_1)^2+(s-a_2)^2$ $+(s-a_3)^2+(s-a_4)^2+\ldots\ldots+(s-a_n)^2=a_1^2+a_2^2+a_3^2+\ldots\ldots$ $+a_n^2$.

53. Shew that $\dfrac{1}{a+x}+\dfrac{2x}{a^2+x^2}+\dfrac{4x^3}{a^4+x^4}+\dfrac{8x^7}{a^8+x^8}=\dfrac{1}{a-x}$ $-\dfrac{16x^{15}}{a^{16}-x^{16}}$.

54. Shew that $(x-y)^4+(y-z)^4+(z-x)^4=2\{(x-y)^2$ $\times(y-z)^2+(y-z)^2(z-x)^2+(z-x)^2(x-y)^2\}=2(x^2+y^2+z^2-xy$ $-zx-yz)^2$.

55. Shew that $(b+c-a)(c+a-b)(a+b-c)+a(b+c-a)^2$ $+b(c+a-b)^2+c(a+b-c)^2=4abc$.

56. Shew that $(a-cx)^2+(x^2-1)(b^2-d^2)$

$$=\left(\frac{ab+b^2+cd-d^2}{c-d}\right)^2 \text{ if } x=\frac{a+b}{c-d}.$$

57. If $s=a+b+c$, then $(as+bc)(bs+ac)(cs+ab)=(a+b)^2$ $\times(b+c)^2(c+a)^2$.

58. If $3s=a+b+c$, $(s-a)^4+(s-b)^4+(s-c)^4=2\{(s-a)^2 \times(s-b)^2+(s-b)^2(s-c)^2+(s-c)^2(s-a)^2\}$.

59. If $\dfrac{1}{1+l+lm}+\dfrac{1}{1+m+mn}+\dfrac{1}{1+n+nl}=1$, prove that—

(i) $lmn=1$, *or* (ii) $(1+l)(1+m)(1+n)=-1$.

60. Shew that $\dfrac{a-b}{n+ab}+\dfrac{b-c}{n+bc}+\dfrac{c-a}{n+ca}$
$$=\frac{n(a-b)(b-c)(c-a)}{(n+ab)(n+bc)(n+ca)}.$$

CHAPTER XV.

EXAMINATION PAPERS.

SECOND SERIES ON CHAPTERS X—XIV.

I.

1. Shew that $(ax+by+cz)^3-(bx+cy+az)^3$ is divisible by $(a-b)x+(b-c)y+(c-a)z$.

2. If $a+b+c+d=0$, shew that $a^3+b^3+c^3+d^3+3(a+b)\times(b+c)(c+a)=0$.

3. Find the H.C.F. of $4x^2+3x-10$ and $4x^3+7x^2-3x-15$.

4. Find the L.C.M. of $x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

5. If x^2-bx+a^2 and x^2-ax+b^2 have a common binomial factor, shew that $2(a+b)^2=ab$.

6. If $x=\dfrac{a+b}{a-b}$ and $y=\dfrac{a-b}{a+b}$, shew that $\dfrac{x+y}{x-y}=\dfrac{a}{2b}+\dfrac{b}{2a}$.

7. Reduce $\dfrac{x^4+x^2y^2+y^4}{x^4+2x^3y+3x^2y^2+2xy^3+y^4}$ to its lowest terms.

8. Simplify $\dfrac{a}{(a-b)(a-c)(a-d)}+\dfrac{b}{(b-a)(b-c)(b-d)}$

$$+\frac{c}{(c-a)(c-b)(c-d)}+\frac{d}{(d-a)(d-b)(d-c)}.$$

9. Shew that $\dfrac{2a-b-c}{(a-b)(a-c)}+\dfrac{2b-a-c}{(b-a)(b-c)}+\dfrac{2c-a-b}{(c-a)(c-b)}=0$.

10. Resolve $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$ into partial fractions.

II.

1. Resolve into factors $(a^2+b^2-c^2)(a^2+c^2-b^2)$
$+(a^2+c^2-b^2)(b^2+c^2-a^2)+(b^2+c^2-a^2)(a^2+b^2-c^2)$.

2. Shew that $4+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{b}{c}+\dfrac{c}{b}\right)\left(\dfrac{c}{a}+\dfrac{a}{c}\right)=\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2$

$$+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2.$$

3. If $a+b+c$ be a factor of $ab+bc+ac$, then it is also a factor of $a^2+b^2+c^2$.

4. If $2s=a+b+c$, prove the following identities :—

(i) $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-a)(s-c)+c(s-a)(s-b)=abc$.

(ii) $s(s-a)(s-b)(s-c)=\frac{1}{16}\{(b+c)^2a^2+(b-c)^2a^2-(b^2-c^2)^2-a^4\}$.

5. If the H.C.F. of two quantities be $x-1$ and their product be $6x^6-7x^5-42x^4+86x^3-42x^2-7x+6$, find their L.C.M.

6. If $\frac{b}{(a+c)^2}=\frac{1}{a+c}-\frac{1}{4b}$, then $b=\frac{a+c}{2}$.

7. Simplify (i) $\frac{(n+1)(2n+1)(2n+3)}{3}-\frac{n(2n-1)(2n+1)}{3}$;

(ii) $\frac{x}{x-1}+\frac{x^2}{x^2+1}-\frac{x^4}{x^4-1}$.

8. If $x=\sqrt{ab}$, shew that $\frac{(x+a)(b-x)}{(x-a)(b+x)}=\frac{(x^2+a^2)(b^2-x^2)}{(x^2-a^2)(b^2+x^2)}$.

9. Find the value of $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}$ if $a+b+c=2$; $ab+ac+bc=1$ and $abc=-1$.

10. Shew that $(x^2-6)^{2n}-(2x^3+x^2-14x+12)^n$ is always divisible by $(x+1)(x-2)(x+3)(x-4)$.

III.

1. What is an identity? Prove the following identities :—

(a) $\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$;

(b) $(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$.

2. Shew that $y^2+2ay-3b^2$ is divisible by $y-a$ if $a+b=0$.

3. How do you find the H.C.F. of two or more fractions?

Find the H.C.F. of $\frac{a^2-bc}{(a-b)(a-c)}$, $\frac{(b^2-ac)}{(b-a)(b-c)}$ and $\frac{c^2-ab}{(c-a)(c-b)}$.

4. Find the H.C.F. of $21x^5-55x^4+1$ and $x^5-55x+21$.

5. If $(a-b)^2+b(a-b)+1=0$, shew that $x+a$ is a common factor of x^2+bx+1 and $x^3+(a+1)x^2+bx+1$.

6. Reduce to its lowest terms $\dfrac{5x^3+2x^2-15x-6}{7x^3-4x^2-21x+12}$.

7. Simplify $\dfrac{a^2(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{b^2(b+a)(b+c)}{(b-a)(b-c)}+\dfrac{c^2(c+a)(c+b)}{(c-a)(c-b)}$.

8. If $x=a(y+z)$, $y=b(z+x)$, $z=c(x+y)$, shew that $\dfrac{a}{1+a}+\dfrac{b}{1+b}+\dfrac{c}{1+c}=1$.

9. Observing that $\dfrac{q-p}{(x-p)(x-q)}=\dfrac{1}{x-q}-\dfrac{1}{x-p}$, simplify

$$\frac{a^2-b^2}{(x^2-a^2)(x^2-b^2)}+\frac{b^2-c^2}{(x^2-b^2)(x^2-c^2)}+\frac{c^2-a^2}{(x^2-c^2)(x^2-a^2)}.$$

10. If $\dfrac{1}{(x+1)(x+2)(x+3)}=\dfrac{p}{x+1}+\dfrac{q}{x+2}+\dfrac{r}{x+3}$, find p, q and r.

IV.

1. State and prove the Rule for finding the L.C.M of two algebraical expressions.

Find the L.C.M. of $12a^3+4a^2-3a-1$ and $8a^3-4a^2-2a+1$.

2. Shew that $a^2x^2+b^2y^2+c^2z^2=2(abxy+bcyz+acxz)$ if $\sqrt{(ax)}+\sqrt{(by)}+\sqrt{(cz)}=0$.

3. Reduce $\dfrac{a^{3m}+a^{2m}-2}{a^{2m}+a^m-2}$ to its lowest terms.

4. Shew that $\dfrac{a^2c^2(a^2-c^2)+b^2c^2(c^2-b^2)+a^2b^2(b^2-a^2)}{ac(a-c)+bc(c-b)+ab(b-a)}$

$$=\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}.$$

5. Find the H.C.F. of $(x+y)^5-x^5-y^5$ and $x^4+x^2y^2+y^4$.

6. Resolve into factors (i) $(1+a)^2(1+b^2)-(1+b)^2(1+a^2)$;
(ii) $x(y^2+z^2-x^2)+y(z^2+x^2-y^2)$.

7. If ax^2-bx+c and dx^2-bx+c have $x-1$ as a common factor, shew that $a^3-abd+cd^2=0$.

8. If $a^3+b^3+c^3=(a+b+c)^3$, then $a^{2n+1}+b^{2n+1}+c^{2n+1}$ $=(a+b+c)^{2n+1}$.

9. If $3(a^2+b^2+c^2)=(a+b+c)^2$, shew that $a=b=c$.

10. Simplify the following complex fractions :—

(i) $\dfrac{x}{y+\dfrac{x}{y+\dfrac{x}{y+\dfrac{x}{y}}}}$ (ii) $\dfrac{x}{x+\dfrac{x}{x+\dfrac{x}{x+\dfrac{x}{x}}}}$.

V.

1. If $x^3+ax^2+(a^2-2ab)x-c^3+3abc$ be divisible by $x+a+b$, shew that $a^3+b^3+c^3=3abc$.

2. Prove that if A measures B and C, it will also measure $mB\pm nC$. Find the H.C.F. of $p^3nq+3np^2q^2-2npq^3-2nq^4$ and $2mp^2q^2-4mp^4-mp^3q+3mpq^3$.

3. If two fractions are equal to 1, shew that their difference is equal to the difference of their squares.

4. Shew that $2^{2n+1}+3^{4n+2}$ is divisible by 11.

5. Resolve (i) $a^4+\dfrac{1}{a^4}+4\left(a^2+\dfrac{1}{a^2}\right)+5$;

(ii) $x^4+2(a+b)x^3+(a^2+4ab+b^2)x^2+2ab(a+b)$ $\times x+a^2b^2$.

6. Simplify (i) $\dfrac{(a+b+c)^3-(a+2c-b)^3-(2b-c)^3}{(a-b+2c)(2b-c)}$;

(ii) $\dfrac{a^4}{(x-a)^n}+\dfrac{4a^3}{(x-a)^{n-1}}+\dfrac{6a^2}{(x-a)^{n-2}}+\dfrac{4a}{(x-a)^{n-3}}$ $+\dfrac{1}{(x-a)^{n-4}}$.

7. If $2s=a+b+c$, shew that $\dfrac{s-a}{(s-b)(s-c)}+\dfrac{s-b}{(s-a)(s-c)}$ $+\dfrac{s-c}{(s-a)(s-b)}+\dfrac{s^2}{(s-a)(s-b)(s-c)}=\dfrac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}$.

8. If $xyz=1$, prove that—

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\times\left(z+\frac{1}{z}\right).$$

9. If $A = ax + by + cz$, $B = cx + ay + bz$ and $C = bx + cy + az$, prove that $(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz) = A^3 + B^3 + C^3 - 3ABC$.

10. Simplify $\dfrac{a^4 + a^2 + 1}{(a^2 + a + 1)(a - b)(a - c)} + \dfrac{b^4 + b^2 + 1}{(b^2 + b + 1)(b - c)(b - a)} + \dfrac{c^4 + c^2 + 1}{(c^2 + c + 1)(c - a)(c - b)}$.

VI.

1. If $(a - b)(b - c) = (c - a)^2$, prove that $(c - a)(a - b) = (b - c)^2$ and $(b - c)(c - a) = (a - b)^2$.

2. Prove that any common multiple of P and Q is a multiple of their L.C.M.

Find by factors the L.C.M. of $12x^2 - 15xy + 3y^2$ and $6x^3 - 6x^2y + 2xy^2 - 2y^3$.

3. The G.C.M. of two numbers is 6, and their L.C.M. is 30. Find the numbers.

4. If the L.C.M. of A and B be equal to the H.C.F. of C and D, shew that $\dfrac{\text{H.C.F. of } A \text{ and } B}{\text{L.C.M. of } C \text{ and } D} = \dfrac{A.B.}{C.D.}$.

5. Simplify (i) $\dfrac{(n+2)^2(n+1)^2 - (n-2)^2(n-1)^2}{(n+1)^3 + n^3 + (n-1)^3}$;

(ii) $\dfrac{(ay - bx)^2 + (ax + by)^2}{\left(\dfrac{a}{b} + \dfrac{b}{a}\right)\left(\dfrac{x}{y} + \dfrac{y}{x}\right)}$.

6. If $a + b + c = 0$, shew that $(ab + ac)(ab + bc) + (ac + ab) \times (ac + bc) + (bc + ab)(bc + ac)$ is a perfect square.

7. Shew that $\dfrac{(a+b)^3 + (b-c)^3}{a - c + 2b} = \dfrac{(a+c)^3 - (b-c)^3}{a + 2c - b}$.

8. Shew that $\dfrac{2}{b-c} + \dfrac{2}{c-a} + \dfrac{2}{a-b}$

$$= \frac{(a-b)^2 + (c-a)^2 + (b-c)^2}{(a-b)(a-c)(b-c)}.$$

9. Find the value of $\dfrac{c}{ax + by}$, when $x = \dfrac{cq - br}{aq - bp}$

and $y = \dfrac{ar - cp}{aq - bp}$.

10. Shew that $\dfrac{x-1}{(n-x)(n-1)}+\dfrac{y-2}{(n-y)(n-2)}+\dfrac{z-3}{(n-z)(n-3)}$

$$=\frac{x-2}{(n-x)(n-2)}+\frac{y-3}{(n-y)(n-3)}+\frac{z-1}{(n-z)(n-1)}.$$

VII.

1. Resolve into factors $(ab+cd)(c^2+d^2)$
$+cd(a^2+b^2-c^2-d^2)$.

2. Divide $x^2(x+y+3)+2(xy+xz)(x+y+3)-x-y-3$
by $x^2+2xz+2xy-1$.

3. Find the H.C.F. of $5x^3-3x^5-2$ and $5x^2-2x^5-3$.

4. Shew that $(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2$
$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

5. If $a+b=cz$, $b+c=ay$, $c+a=bx$, shew that $\dfrac{1}{x+1}+\dfrac{1}{y+1}$
$+\dfrac{1}{z+1}=1$.

6. Shew that $x(x-2a)(x+a)(x-a)+a^4$ is a perfect square.

7. Find the L.C.M. of $a^{11}+a^4$ and $a^{10}+a^7$.

8. Reduce $\dfrac{a^4-5a^2+4}{a^2-3a+2}$ to its lowest terms.

9. Simplify (i) $\dfrac{(2y-x)^3-(2x-y)^3}{3(y-x)}+\dfrac{(2y-x)^3+(2x-y)^3}{y+x}$;

(ii) $\dfrac{\dfrac{p}{q+r}+\dfrac{q}{r+p}+\dfrac{r}{p+q}+3}{\dfrac{1}{q+r}+\dfrac{1}{r+p}+\dfrac{1}{p+q}}$; (iii) $\dfrac{\dfrac{a}{2b-a}+\dfrac{b}{2a-b}}{\dfrac{3ab}{(2b-a)(2a-b)}-1}$.

10. If $a+b+c=0$, shew that—
$(ab+ac+bc)^3+(a^2-bc)(b^2-ac)(c^2-ab)=0$.

VIII.

1. Shew that $x(y-z)^3+y(z-x)^3+z(x-y)^3=(x-y)(y-z)$
$\times(z-x)(x+y+z)$.

2. If $x+p$ be the H.C.F. of x^2+ax+1 and x^2+bx+2,
then $p=\dfrac{1}{b-a}$.

3. State and prove the rule for finding the H.C.F. of two algebraical expressions.

4. If $ax^2+bxy+cy^2$ and $bx^2-2(a-c)xy-by^2$ have a common factor, then $ax^2+bxy+cy^2$ is an exact square.

5. If $x+y+z=2a$ and $x^2+xy+y^2+a^2=2a(x+y)$, shew that $(x-a)^2+(y-a)^2+(z-a)^2=a^2$.

6. Simplify :—

(i) $\dfrac{a+b}{(a^2-bc)(b^2-ac)}+\dfrac{b+c}{(b^2-ac)(c^2-ab)}+\dfrac{c+a}{(c^2-ab)(a^2-bc)}$;

(ii) $\dfrac{a^2(c-b)}{(a+b)(a+c)}+\dfrac{b^2(a-c)}{(b+c)(b+a)}+\dfrac{c^2(b-a)}{(c+a)(c+b)}$.

7. If $\dfrac{(a+bc)^2}{(b+ac)^2}=\dfrac{1-b^2}{1-a^2}$, then $a^2+b^2+c^2+2abc=1$.

8. If $(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd)$, shew that $a=b=c=d$.

9. Shew that px^4+qx^2+r is divisible by x^2+r, if $q=pr+1$.

10. Decompose $\dfrac{7x+11}{(x+1)(x+2)(x+3)}$ into the sum of three partial fractions.

IX.

1. Find the relation between a and b in order that $y^5-5ay+4b$ may be exactly divisible by $(y-b)^2$.

2. Shew that $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}\right)^2-\dfrac{(a+b+c+d)^2}{abcd}=\left(\dfrac{1}{ad}-\dfrac{1}{bc}\right)$ $\times\left(\dfrac{1}{bd}-\dfrac{1}{ac}\right)\left(\dfrac{1}{cd}-\dfrac{1}{ab}\right)abcd.$

3. The H.C.F. of two expressions is $a-4$, and their L.C.M. is $a^4-10a^3+35a^2-50a+24$; find the expressions.

4. If $a^3(b-c)+b^3(c-a)+c^3(a-b)=(a-b)(a-c)(b-c)$ $\times(pa+qb+rc)$, find p, q and r.

5. Reduce $\dfrac{9x^3+53x^2-9x-18}{4x^2+44x+120}$ to its lowest terms.

6. Simplify (i) $\dfrac{(a+b)^3+(b+c)^3-(a+2b+c)^3}{(a+b)(b+c)(a+2b+c)}$;

(ii) $\dfrac{b-c}{x-a}+\dfrac{c-a}{x-b}+\dfrac{a-b}{x-c}+\dfrac{(a-b)(b-c)(c-a)}{(x-a)(x-b)(x-c)}$.

7. If x^2+ax+b, and $x^2+a'x+b'$ have a common factor of the first degree, shew that their L.C.M. is $x^3+\dfrac{ab-a'b'}{b-b'}x^2$ $+\left\{aa'-\left(\dfrac{b-b'}{a-a'}\right)\right\}x+bb'\dfrac{a-a'}{b-b'}$.

8. Shew that $(b+c-a)(c+a-b)(a+b-c)=a^2(b+c)$ $+b^2(c+a)+c^2(a+b)-a^3-b^3-c^3-2abc$.

9. Find the value of $\left(\dfrac{x}{x-1}\right)^2+\left(\dfrac{x}{x+1}\right)^2$ when $x^2=\dfrac{n-1}{n+1}$.

10. If $a^2+c^2=2b^2$, shew that $\dfrac{1}{b+c}+\dfrac{1}{a+b}=\dfrac{2}{a+c}$.

X.

1. How do you find the H.C.F. of *more than two* expressions? Find the H.C.F. of x^4-6x^2+8x-3, $x^4-2x^3-7x^2+20x-12$ and $x^4-4x^2+12x-9$.

2. Shew that $x^n-1-n(x-1)$ is divisible by $(x-1)^2$.

3. Reduce $\dfrac{8x^7-377x^4+21}{21x^7-377x^3+8}$ to its lowest terms.

4. Simplify $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3$ $-(x+y-z)^3$.

5. If $(a+b+c)^3=a^3+b^3+c^3$, prove that $(a+b+c)^5=a^5+b^5+c^5$.

6. If $a^2+b^2+c^2=0$, prove that $(a^2+ab+b^2)^3$ $+(a^2-ab+b^2)^3-6c^2(a^4+a^2b^2+b^4)=-8c^6$.

7. Resolve into factors (i) $x^3-3abx+(a+b)(a^2-ab+b^2)$;
(ii) $x^3-3ax^2+3(a^2-bc)x-(a^3+b^3+c^3-3abc)$.

8. If $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}+\dfrac{1}{d}$, shew that $\dfrac{a^2(b-c)}{a-d}=\dfrac{b^2(a-c)}{b-d}$.

9. Simplify (i) $$\frac{x^2\left(\frac{1}{y}-\frac{1}{z}\right)+y^2\left(\frac{1}{z}-\frac{1}{x}\right)+z^2\left(\frac{1}{x}-\frac{1}{y}\right)}{x\left(\frac{1}{y}-\frac{1}{z}\right)+y\left(\frac{1}{z}-\frac{1}{x}\right)+z\left(\frac{1}{x}-\frac{1}{y}\right)};$$

(ii) $$\frac{1}{1+x+x^2}+\frac{1}{1+x+\frac{1}{x}}+\frac{1}{1+\frac{1}{x}+\frac{1}{x^2}};$$

(iii) $$\frac{\frac{x+y}{x-y}+\frac{y+z}{y-z}+\frac{z+x}{z-x}+3}{\frac{x}{x-y}+\frac{y}{y-z}+\frac{z}{z-x}}.$$

10. If $x+y+z=0$, shew that $x^5+y^5+z^5=-5xyz\times(xy+yz+zx)$ and $x^6+y^6+z^6=3x^2y^2z^2-2(xy+yz+zx)^3$.

CHAPTER XVI.

INVOLUTION.

108. If a quantity be continually multiplied by itself, it is said to be *involved* or raised; and the power to which it is raised is expressed by the number of times the quantity has been employed in multiplication.

Thus, $a \times a$ or a^2 is the second power of a; $a \times a \times a$ or a^3 is the third power of a; and so on.

If the quantity to be *involved* has a negative sign, the sign of the *even* powers will be positive, and the sign of the *odd* powers will be negative.

Thus, $-a \times -a = a^2$; $-a \times -a \times -a = -a^3$;
$-a \times -a \times -a \times -a = a^4$; and so on.

109. A *simple* quantity is raised to any power by multiplying the index of every factor in the quantity by the exponent of that power, and prefixing the proper sign.

Thus, $(a^m)^n = a^{mn}$; $(ab)^m = ab \times ab \times ab \ldots\ldots$ to m factors, $= a \times a \times a \ldots\ldots$ to m factors $\times b \times b \times b \ldots\ldots$ to m factors $= a^m \times b^m$.

$(a^2b^4c)^5 = a^{10}b^{20}c^5$. And $(-a^n)^m = +a^{nm}$ if m is *even*; or $= -a^{mn}$ if m is *odd*.

110. To raise a Binomial to any power.

By actual multiplication, the following results can be established.

$$\left.\begin{array}{l}(a+b)^2 = a^2 + 2ab + b^2 \\ (a-b)^2 = a^2 - 2ab + b^2\end{array}\right\} \ldots\ldots\ldots\ldots \text{I.}$$

$$\left.\begin{array}{l}(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\ (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\end{array}\right\} \ldots\ldots\ldots\ldots \text{II.}$$

$$\left.\begin{array}{l}(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{array}\right\} \ldots\ldots \text{III.}$$

$$\left.\begin{array}{l}(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\ (a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5\end{array}\right\} \ldots \text{IV.}$$

On examining the above cases we observe:—

(1) The number of terms in the resulting expression is one more than the index of the power.

(2) Any power of $a-b$ differs from that of $a+b$ *only* in this, that the signs of the terms of the former are *alternately* + and —, whilst those of the latter are *all* +.

(3) The first term of the expansion is equal to the first term of the binomial raised to the given power. In the succeeding terms, the powers of a decrease by unity, and those of b increase by unity.

(4) The co-efficient of the second term is the same as the index of the power to which the binomial is raised.

(5) If we multiply the co-efficient of any term by the power of a in that term, and divide the product by the number of that term, we get the co-efficient of the following term.

(6) The co-efficient of the terms equidistant from the beginning and the end are equal.

The above laws enable us to expand a binomial raised to any power.

Ex. 1. Expand $(a+b)^6$ and $(a-b)^6$.

The *first* term $=a^6$. The *second* term $=6a^5b$.

The *third* term $=\dfrac{6\times5}{2}a^4b^2=15a^4b^2$.

The *fourth* term $=\dfrac{15\times4}{3}a^3b^3=20a^3b^3$.

The *fifth* term $=\dfrac{20\times3}{4}a^2b^4=15a^2b^4$.

The *sixth* term $=\dfrac{15\times2}{5}ab^5=6ab^5$.

The *seventh* term $=\dfrac{6\times1}{6}b^6=b^6$.

Hence, $(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$ and $(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.

Ex. 2. Expand $(x-2y)^7$.

The *first* term $=x^7$.

The *second* term $=-7x^6(2y)=[-14x^6y]$.

The *third* term $=+\dfrac{7\times6}{2}x^5(2y)^2=21x^5(2y)^2=[84x^5y^2]$.

The *fourth* term $= -\frac{21\times5}{3}x^4(2y)^3 = -35x^4(2y)^3$
$= [-280x^4y^3]$.

The *fifth* term $= +\frac{35\times4}{4}x^3(2y)^4 = 35x^3(2y)^4 = [560x^3y^4]$.

The *sixth* term $= -\frac{35\times3}{5}x^2(2y)^5 = -21x^2(2y)^5$
$= [-672x^2y^5]$.

The *seventh* term $= +\frac{21\times2}{6}x(2y)^6 = 7x(2y)^6 = [448xy^6]$.

The *eighth* or last term $= -\frac{7\times1}{7}(2y)^7 = -(2y)^7$
$= [-128y^7]$.

Hence, $(x-2y)^7 = x^7 - 14x^6y + 84x^5y^2 - 280x^4y^3 + 560x^3y^4 - 672x^2y^5 + 448xy^6 - 128y^7$.

Ex. 3. Find the co-efficient of x^5 in the expansion of $(x+a)^8$.

$(x+a)^8 = x^8 + 8x^7a + \left(\frac{8\times7}{2}\text{ or }28\right)x^6a^2 + \left(\frac{28\times6}{3}\text{or }56\right)x^5a^3 + \&c.$ Hence the co-efficient of $x^5 = 56a^3$.

N.B.—A trinomial raised to any power may be expanded by the repeated application of this rule.

EXERCISE 45.

Expand :—

1. $(x-a)^5$. 2. $(a^2+1)^5$. 3. $(a^3-2)^4$. 4. $(x-y)^6$.
5. $(2-x)^7$. 6. $\left(1+\frac{1}{x}\right)^6$. 7. $(4+a^3)^4$. 8. $(3x-2a)^5$.
9. $(1+a+a^2)^4$. 10. $(x-1+x^2)^5$. 11. $(ax+by)^5$.
12. $(ax^2+bx+c)^3$.

Simplify :—

13. $(a+b)^6-(a-b)^6$. 14. $(x+y)^7+(x-y)^7$.
15. $(a+b)^5-a^5-b^5$. 16. $(a+b)^7-a^7-b^7$.

Find the co-efficient of :—

17. x^4 in $(2x+a)^5$. 18. x^5 in $(x-2y)^6$.
19. x^2 in $(x-3a)^7$. 20. x^3 in $(x^2+x+1)^4$.
21. x^4 in $(1-x)^4(1+x)^4$. 22. x^5 in $(1-x)^3(1+x)^3$.

CHAPTER XVII.

EVOLUTION.

111. Evolution, or the extraction of roots, is the reverse of involution and it is the method of determining a quantity which raised to a proposed power will produce the given quantity.

To find the root of a simple quantity.—" Divide the exponent of each factor by the index of the required root, and prefix the root of the numerical co-efficient." Thus, $\sqrt[5]{a^{25}}=a^5$; $\sqrt[3]{27a^9}=3a^3$; $\sqrt[4]{16a^8b^4c^{12}}=2a^2bc^3$.

Sign of Roots.—The root of a positive quantity is positive or negative if the index of the root be an even number.

Thus, $\sqrt{a^2}=\pm a$; $\sqrt[4]{16a^8}=\pm 2a^2$.

The root of a negative quantity is negative if the index of the root be an odd number. Thus, $\sqrt[3]{-a^3}=-a$.

The root of a positive quantity is positive if the index of the root be an odd number. Thus, $\sqrt[3]{a^6}=a^2$.

The root of a negative quantity is impossible when the index of the root is an even number. Thus, $\sqrt{-2}$ is an imaginary quantity.

112. The ordinary method of extracting the square root of a compound expression.

Rule.—" Arrange the given expression according to ascending or descending powers of some contained letter.

Find the square root of the first term, which is the first term of the required root; subtract its square from the given expression, and divide the first term of the remainder by double the part of the root already found, and the quotient is the second term of the required root.

Double the first term of the root, and add the second term; multiply the sum by the second term, and subtract the product from the remainder; if there is no remainder, the square root is found. If there is a remainder we divide the first term of it by double the part of the root already found: the quotient gives the third term of the root, and so on."

The rule is deduced from the following :—

$$(a+b)^2=a^2+(2a+b)b.$$
$$(a+b+c)^2=a^2+(2a+b)b+(2a+2b+c)c.$$

$$(a+b+c+d)^2=a^2+(2a+b)b+(2a+2b+c)c$$
$$+(2a+2b+2c+d)d.$$

Ex. 1. Extract the square root of $4a^2+b^2+4ab$.

Arrange the expression according to powers of a.

$$\begin{array}{ll}
2a) & 4a^2+4ab+b^2(2a+b \\
 & 4a^2 \\
4a+b) & \quad 4ab+b^2 \\
 & \quad 4ab+b^2
\end{array}$$

The square root of $4a^2$ is $2a$, which is the first term of the required root We place it to the right of the given expression and also to the left. We now subtract the square of $2a$ from the given expression, and the remainder is $4ab+b^2$; then we double $2a$, which gives $4a$, by which we divide the first term of the remainder, and this gives b as quotient, which is the second term of the square root We place it after $2a$ to the right; and we also add it to $4a$ Again we multiply $4a+b$ by b, the second term of the root, and subtract the product from the remainder. Since there is no remainder, we have $2a+b$ for the required square root.

Ex. 2. Extract the square root of $x^4-2x^3+3x^2-2x+1$

The expression is arranged according to powers of x.

$$\begin{array}{ll}
 & x^4-2x^3+3x^2-2x+1(x^2-x+1. \\
 & x^4 \\
2x^2-x & \quad -2x^3+3x^2-2x+1 \\
 & \quad -2x^3+\ \ x^2 \\
2x^2-2x+1 & \qquad\quad 2x^2-2x+1 \\
 & \qquad\quad 2x^2-2x+1
\end{array}$$

Hence the required square root is x^2-x+1.

Ex. 3. Extract the square root of—

$$x^6-6ax^5+15a^2x^4-20a^3x^3+15a^4x^2-6a^5x+a^6.$$

Arrange the expression according to descending powers of a.

$$\begin{array}{ll}
a^3) & a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6(a^3 \\
 & a^6 \qquad\qquad\qquad\qquad\qquad\qquad\quad -3a^2x+3ax^2-x^3 \\
2a^3-3a^2x) & -6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6 \\
 & -6a^5x+\ \ 9a^4x^2 \\
2a^3-6a^2x+3ax^2) & \qquad 6a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6 \\
 & \qquad 6a^4x^2-18a^3x^3+\ \ 9a^2x^4 \\
2a^3-6a^2x+6ax^2-x^3) & \qquad\qquad -2a^3x^3+\ \ 6a^2x^4-6ax^5+x^6 \\
 & \qquad\qquad -2a^3x^3+\ \ 6a^2x^4-6ax^5+x^6
\end{array}$$

Hence the required square root is $a^3-3a^2x+3ax^2-x^3$.

Ex. 4. Extract the square root of $1-\frac{a}{2}+\frac{37a^2}{80}-\frac{a^3}{10}+\frac{a^4}{25}$.

The expression is arranged according to ascending powers of a.

$$\begin{array}{l|l}
1 & 1-\frac{a}{2}+\frac{37a^2}{80}-\frac{a^3}{10}+\frac{a^4}{25} \quad \left(\, 1-\frac{a}{4}+\frac{a^2}{5}\right. \\
 & 1 \\ \hline
2-\frac{a}{4} & -\frac{a}{2}+\frac{37a^2}{80}-\frac{a^3}{10}+\frac{a^4}{25} \\
 & -\frac{a}{2}+\frac{a^2}{16} \\ \hline
2-\frac{a}{2}+\frac{a^2}{5} & \frac{2a^2}{5}-\frac{a^3}{10}+\frac{a^4}{25} \\
 & \frac{2a^2}{5}-\frac{a^3}{10}+\frac{a^4}{25}
\end{array}$$

Hence $1-\frac{a}{4}+\frac{a^2}{5}$ is the required square root.

Ex. 5. Extract the square root of :—

$$x^4+\frac{1}{x^4}+x^2+\frac{1}{x^2}+2\left\{x^3-\frac{1}{x^3}-\left(x-\frac{1}{x}\right)\right\}.$$

Arrange the expression according to descending powers of x.

$$\begin{array}{l|l}
x^2 & x^4+2x^3+x^2-2x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{1}{x^4} \quad \left(\, x^2+x-\frac{1}{x}+\frac{1}{x^2}\right. \\
 & x^4 \\ \hline
2x^2+x & 2x^3+x^2-2x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{1}{x^4} \\
 & 2x^3+x^2 \\ \hline
2x^2+2x-\frac{1}{x} & -2x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{1}{x^4} \\
 & -2x-2+\frac{1}{x^2} \\ \hline
2x^2+2x-\frac{2}{x}+\frac{1}{x^2} & 2+\frac{2}{x}-\frac{2}{x^3}+\frac{1}{x^4} \\
 & 2+\frac{2}{x}-\frac{2}{x^3}+\frac{1}{x^4}
\end{array}$$

The square root is $x^2+x-\frac{1}{x}+\frac{1}{x^2}$.

Ex. 6. Extract the square root of :—

$$x^2-2+\frac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}.$$

In such examples as this, we must first see *whether the fraction is in its lowest terms.*

Now the H.C.F. of $4x^3+9x^2+4x+9$ and $x^4+4x^3+5x^2+4x+4$ is x^2+1, therefore $\frac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}$

$$=\frac{(4x+9)(x^2+1)}{(x^2+4x+4)(x^2+1)}=\frac{4x+9}{x^2+4x+4}.$$

The given expression $=x^2-2+\dfrac{4x+9}{x^2+4x+4}$

$$=\frac{x^4+4x^3+2x^2-4x+1}{x^2+4x+4}.$$

The square root of the numerator is x^2+2x-1.

The square root of the denominator is $x+2$.

$\therefore$ The square root of the given expression $=\dfrac{x^2+2x-1}{x+2}$

$$=x-\frac{1}{x+2}.$$

EXERCISE 46.

Extract the square roots of

1. $81a^2b^2+18abc^2+c^4$.
2. $49a^2-42ab+9b^2$.
3. $a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2$.
4. $9-6x+13x^2-4x^3+4x^4$.
5. $(2a^2b+b^2)^2+(a^2-2ab^2)(4ab+1)$.
6. $x^6+8x^4-2x^3+16x^2-8x+1$.
7. $\dfrac{x^4}{4}+4x^2+\dfrac{ax^2}{3}+\dfrac{a^2}{9}-2x^3-\dfrac{4ax}{3}$.

8. $9x^4 - 2x^3y + \frac{163}{9}x^2y^2 - 2xy^3 + 9y^4$.

9. $x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}$.

10. $\frac{x^4}{4y^4} + \frac{4y^4}{x^4} + \frac{x^2}{y^2} + \frac{4y^2}{x^2} + 3$. 11. $\frac{x^2}{y^2} + \frac{y^2}{4x^2} - \frac{x}{y} + \frac{y}{2x} - \frac{3}{4}$.

12. $x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 3$.

13. $\frac{1}{a^4} - \frac{1}{a^2} + \frac{2}{a^3} + 3 - 2a + a^2$

14. $(a-b)^4 - 2(a^2+b^2)(a-b)^2 + 2(a^4+b^4)$.

15. $a^4 + b^4 + c^4 + d^4 - 2a^2(b^2+d^2) - 2b^2(c^2-d^2)$
$+ 2c^2(a^2-d^2)$.

16. $a^4 + 2(2b-c)a^3 + (4b^2-4bc+3c^2)a^2 + 2c^2(2b-c)a + c^4$.

17. $\left(a + \frac{1}{a}\right)^2 - 4\left(a - \frac{1}{a}\right)$.

18. $a^2 + 3 + \frac{6(a^2-1)}{4a^3-8a^2+5a-1}$.

19. $x^2 + 2x + \frac{8-16x+6x^2}{2-9x+12x^2-4x^3}$.

20. $a^2 + 2 - \frac{28a^2+45a-7}{7a^3+27a^2+24a-4}$.

113. *When* n+1 *figures of a square root have been found by the ordinary method,* n *more may be obtained by division only, supposing* 2n+1 *to be the whole number of figures in the root.*

Let N represent the number whose square root is required; a the part of the root already found; and x the part to be found; then, $\sqrt{N} = a + x$. $\therefore N = a^2 + x^2 + 2ax$. $\therefore N - a^2 = 2ax + x^2$.

$\therefore \frac{N-a^2}{2a} = x + \frac{x^2}{2a}$. If we show that $\frac{x^2}{2a}$ is a *proper fraction* then x = the quotient of $N - a^2$ divided by $2a$.

Since x contains n digits, x^2 cannot contain more than $2n$ digits; but a contains $2n+1$ digits. Therefore $\frac{x^2}{2a}$ is a proper fraction.

114. Miscellaneous Examples.

Ex. 1. To find the square root of $x^4+4x^3+10x^2+12x+9$ by the *Method of Indeterminate Co-efficients.*

Let $x^4+4x^3+10x^2+12x+9=(x^2+px+q)^2$; then $x^4+4x^3+10x^2+12x+9=x^4+2px^3+(p^2+2q)x^2+2pqx+q^2$.

Since this is true for all values of x, the co-efficients of the like powers of x, are equal.

Hence, $2p=4 \therefore p=2$; $p^2+2q=10$; $\therefore 2q=10-4=6 \therefore q=3$ $\therefore$ the square root of the given expression is x^2+2x+3.

Ex. 2. To find the square root of $4a^2+b^2+9c^2+6bc-12ac-4ab$ by reducing it to the form $(A \pm B)^2$.

The given expression $=(4a^2+b^2-4ab)+9c^2+6bc-12ac$
$=(2a-b)^2+9c^2-6c(2a-b)$
$=(2a-b)^2+(-3c)^2+2(-3c)(2a-b)=(2a-b-3c)^2$.

Hence, the square root is $2a-b-3c$.

Ex. 3. What value of x will make $x^4+4x^3+10x^2+14x+3$ a *perfect* square?

Rule.—"Extract the square root in the ordinary way and put the remainder $=0$."

$$
\begin{array}{r|l}
x^2) & x^4+4x^3+10x^2+14x+3(x^2+2x+3 \\
 & x^4 \\
\hline
2x^2+2x & 4x^3+10x^2+14x+3 \\
 & 4x^3+4x^2 \\
\hline
2x^2+4x+3 & 6x^2+14x+3 \\
 & 6x^2+12x+9 \\
\hline
 & 2x-6
\end{array}
$$

Putting $2x-6=0$, we have $2x=6 \therefore x=3$.

Ex. 4. The remainder after finding the first two terms of a square root of the form ax^2+bx+c is $-6x^2+4x+1$; determine the root.

$(ax^2+bx+c)^2$ is the whole expression of which the square root is ax^2+bx+c and $(ax^2+bx)^2$ is that portion of the expression which is necessary for finding the first two terms of the root.

$$\therefore \quad (ax^2+bx+c)^2-(ax^2+bx)^2=-6x^2+4x+1.$$

$$\therefore \quad (2ax^2+2bx+c)c=-6x^2+4x+1.$$

$$\therefore \quad 2acx^2+2bcx+c^2=-6x^2+4x+1.$$

Since this is true for all values of x, the co-efficients of the like powers of x are equal.

$$\therefore \quad c^2=1 \therefore c=\pm 1;\ 2bc=4 \therefore b=\pm 2;\ 2ac=-6$$

$$\therefore \quad a=\mp 3 \therefore \text{ the root is } -3x^2+2x+1 \text{ or } 3x^2-2x-1,$$

or thus: The form of the trial divisor after finding the first two terms must be $2ax^2+2bx$.

$\therefore$ the form of the remainder must be $(2ax^2+2bx+c)c$.

$\therefore$ $(2ax^2+2bx+c)c=-6x^2+4x+1$. Hence a, b and c can be found.

Ex. 5. Find the relation between a, b and c in order that ax^2+bx+c may be an *exact* square.

If ax^2+bx+c be an exact square, it must be the square of a binomial of the form $x\sqrt{a}+\sqrt{c}$.

$$\therefore \quad ax^2+bx+c=(x\sqrt{a}+\sqrt{c})^2=ax^2+2x\sqrt{ac}+c.$$

$$\therefore \quad b=2\sqrt{ac} \therefore b^2=4ac.$$

Find the relation between a, b, c and d, in order that $x^4+ax^3+bx^2+cx+d$ may be a *perfect* square.

If the given expression be an exact square, it must be the square of a trinomial of the form $x^2+px+\sqrt{d}$.

$\therefore$ $x^4+ax^3+bx^2+cx+d=(x^2+px+\sqrt{d})^2=x^4+2px^3+(p^2+2\sqrt{d})x^2+2xp\sqrt{d}+d$.

Equating the co-efficients of the like powers of x, $a=2p$; $b=p^2+2\sqrt{d}$; $c=2p\sqrt{d}$. $\therefore$ $p=\frac{a}{2}$; $b=\frac{a^2}{4}+2\sqrt{d}$ and $c=a\sqrt{d}$.

$$\therefore \quad 4b-a^2=8\sqrt{d} \text{ and } c^2=a^2d,$$

or thus: Extract the square root in the ordinary way thus:

$$x^2\Big)\, x^4+ax^3+bx^2+cx+d\ \Big(x^2+\frac{a}{2}x+\frac{b-\frac{a^2}{4}}{2}$$

$$x^4$$

$$2x^2+\frac{a}{2}x\Big)\, ax^3+bx^2+cx+d$$

$$ax^3+\frac{a^2}{4}x^2$$

$$2x^2+ax+\frac{b-\frac{a^2}{4}}{2}\Big)\, x^2\left(b-\frac{a^2}{4}\right)+cx+d$$

$$x^2\left(b-\frac{a^2}{4}\right)+\frac{a}{2}\left(b-\frac{a^2}{4}\right)x+\frac{1}{4}\left(b-\frac{a^2}{4}\right)^2$$

$$x\left\{c-\frac{a}{2}\left(b-\frac{a^2}{4}\right)\right\}+d-\frac{1}{4}\left(b-\frac{a^2}{4}\right)^2.$$

Now this remainder must vanish identically if the given expression is a perfect square. It will vanish if

$$c-\frac{a}{2}\left(b-\frac{a^2}{4}\right)=0 \text{ and, } d-\frac{1}{4}\left(b-\frac{a^2}{4}\right)^2=0.$$

$$\therefore\ d=\frac{1}{4}\left[b-\frac{a^2}{4}\right]^2 \quad \therefore \sqrt{d}=\frac{1}{2}\left[b-\frac{a^2}{4}\right] \quad \therefore 8\sqrt{d}=4b-a^2.$$

$$\therefore c-a(\sqrt{d})=0 \qquad \therefore\ c=a\sqrt{d} \quad \therefore c^2=a^2d.$$

Ex. 6. Extract the square root of $1-x$ to 4 terms.

$$1\Big]\, 1-x\Big[\,1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}$$

$$1$$

$$2-\frac{x}{2}\Big]-x$$

$$-x+\frac{x^2}{4}$$

$$2-x-\frac{x^2}{8}\Big]-\frac{x^2}{4}$$

$$-\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{64}.$$

$$2-x-\frac{x^2}{4}-\frac{x^3}{16}\Big]-\frac{x^3}{8}-\frac{x^4}{64}$$

$$-\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{64}+\frac{x^6}{256}$$

$$-\frac{5x^4}{64}-\frac{x^5}{64}-\frac{x^6}{256}.\ \text{(Remdr.)}$$

Hence $\sqrt{(1-x)} = 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\&c.$ $\therefore 1-x$

$$= \left[1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\right]^2 - \frac{5x^4}{64} - \frac{x^5}{64} - \frac{x^6}{256}.$$

Changing the sign of x, we have $\sqrt{(1+x)} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \&c.$

Ex. 7. The product of any four numbers which increase or decrease by a common difference together with the fourth power of that difference is a perfect square.

$(a+x)(a+2x)(a+3x)(a+4x)+x^4$ shall be a perfect square.

Expression $= \{(a+x)(a+4x)\}\{(a+2x)(a+3x)\}+x^4$

$= (a^2+5ax+4x^2)(a^2+5ax+6x^2)+x^4$

$= p(p+2x^2)+x^4$; putting p for $a^2+5ax+4x^2$

$= p^2+2px^2+x^4 = (p+x^2)^2 = (a^2+5ax+4x^2+x^2)^2$

$= (a^2+5ax+5x^2)^2$.

Hence $x(x+1)(x+2)(x+3)+1$ and $(x-2)x(x+2)(x+4)+16$ are perfect squares.

I.E.—The product of any four consecutive numbers together with unity is a perfect square ; and the product of any four consecutive odd or even numbers together with 16 is a perfect square.

Ex. 8. If three numbers increase or decrease by a common difference, then the product of the first and the third together with the square of the common difference = square of the second.

$(a+x)(a+3x)+x^2 = (a+2x)^2$. Now $(a+x)(a+3x)+x^2$ $= a^2+4ax+3x^2+x^2 = a^2+4ax+4x^2 = (a+2x)^2$.

Hence $(a+1)(a+3)+1 = (a+2)^2$; and $(a-2)(a-6)+4$ $= (a-4)^2$; and $(a-b)(a-5b)+4b^2 = (a-3b)^2$.

EXERCISE 47.

Find, *by the Method of Indeterminate Co-efficients*, the square roots of—

1. $9x^4-12x^3+10x^2-4x+1$. 2. $x^4-\frac{1}{2}x+\frac{3}{2}x^2+\frac{1}{16}-2x^3$.

3. $x^2+\frac{2ax}{3}-bx+\frac{a^2}{9}+\frac{b^2}{4}-\frac{ab}{3}$.

4. $x^6+6x^5y+15y^2x^4+6xy^5+20x^3y^3+y^6+15x^2y^4$.

5. $x^4-6x^3y+13x^2y^2-12xy^3+4y^4$.

6. $x^8-2a^2x^6-a^4x^4+2a^6x^2+a^8$.

7. $9\frac{a^4}{b^4}+24\frac{a^3}{b^3}+34\frac{a^2}{b^2}+12\frac{a}{b}-7-12\frac{b}{a}+4\frac{b^2}{a^2}$.

8. $16a^4+16a^3-4a^2-28a-11+\frac{6}{a}-\frac{9}{a^2}$.

9. $4(x-2)^4+20(x-2)^3+(x-2)^2-60(x-2)+36$.

10. $(a-1)^4+6(a-1)^3+5(a-1)^2-12(a-1)+4$.

Find the square roots of the following reducing each to be form $(A\pm B)^2$:—

11. $a^2+b^2+c^2+2ab-2ac-2bc$.

12. $a^4+4b^4+c^4-4a^2b^2+2a^2c^2-4b^2c^2$.

13. $\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)$. 14. $a^4+\frac{1}{a^4}+2\left(a^2+\frac{1}{a^2}\right)+3$.

15. $\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)+6$. 16. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{2x}{y}+\frac{2y}{x}+3$.

For what value of x will each of the following expressions be an *exact* square?—

17. $x^4+6x^3+11x^2+3x+31$. 18. $x^4+4x^3+2x^2-3x-1$.

19. $x^4+4bx^3+2b^2x^2-3b^3x-a^4$.

20. $4x^4+24x^3+44x^2+12x+124$.

21. $25x^4-30ax^3+49a^2x^2-22a^3x+4a^4$.

22. $x^4+6x^3+10x^2+x+20$.

23. The remainder after finding the first two terms of a square root of the form ax^2+bx+c is $4x^2-2x+1$. Find the whole root.

24. The remainder after finding the first two terms of a square root of the form px^2+qx+r is $6x^2+18x+9$; determine the whole root.

25. The remainder after finding the first two terms of a square root of the form $ax^2 + bx + c$ is $12x^2 + 12x + 9$. Find the whole root.

26. The remainder after finding the first three terms of a square root of the form $pa^3 + qa^2 + ra + s$ is $6a^3 + 6a^2 + 18a + 9$. Find the whole root.

27. Find the relation between a, b, c and d, in order that $x^4 + 4ax^3 + 6bx^2 + 4cx + d$ may be a perfect square.

28. Find the relation between p and q in order that $x^2 + px + q$ may be a perfect square.

29. Find the relation between p, q, r and s, in order that $x^4 + px^3 + qx^2 + rx + s$ may be a perfect square.

30. Find the relation between p and q, in order that $x^4 + 6p^2q^2x^3 + (p^2 + q^2)x^2 + p^6x + p^2q^2$ may be a perfect square.

Find the square root of each of the following to 5 terms :—

31. $1 - a^2$; $1 + a^2$.

32. $a^2 - x^2$; $a^2 + x^2$.

33. $1 + x + x^2$; $1 - x + x^2$.

34. $a + b$; $a - b$.

Shew that each of the following is a perfect square :—

35. $4(x+1)(x+2)(2x+1)(2x+3) + 1$; $(x^2 + x + 1) \times (x^2 + 3x + 3) + (x+1)^2$.

36. $(x+2)(2x+3)(3x+4)(4x+5) + (x+1)^4$; $(x - a - b) \times (x - 3a - 5b) + (a + 2b)^2$.

37. $(a^2 + 2a + 2)(a^2 - 2a + 2)(a^4 + 3)(a^4 + 2)(a^4 + 1) + 1$.

38. $(x + a + b)(x + a + 3b)(x + a - b)(x + a - 3b) + 16b^4$.

39. $(a-b)^2(b-c)^2 + (b-c)^2(c-a)^2 + (c-a)^2(a-b)^2$.

40. $2\{(x+y)^4 + (2y-x)^4 + (2x-y)^4\}$.

115. The ordinary method of extracting the cube root of a compound expression.

Rule.—"Arrange the expression according to ascending or descending powers of some contained letter.

Find the cube root of the first term, which is the first term of the required cube root; subtract its cube from the expression, and divide the first term of the remainder by three times the square of the root already found, and the quotient is the second term of the required root,

Add together three times the square of the first term of the root, three times the product of its first and second terms, and the square of the second term; multiply the sum by the second term of the root and subtract the product from the remainder already found. If there is no remainder, we have the cube root required. If there is a remainder, we divide it by three times the square of the two terms of the root already found, and the quotient is the third term of the root. Now we treat the first two terms as if it were the first term, and the third term as if it were the second, and proceed as before; and so on till we find no remainder, or as many terms of the root as we may require."

The rule is deduced from the following:—

$$(a+b)^3=a^3+(3a^2+3ab+b^2)b.$$

$$(a+b+c)^3=a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c.$$

Ex. 1. Extract the cube root of $8a^3-36a^2+54a-27$.

The expression is arranged according to descending powers of a.

$$\begin{array}{l|l}
 & 8a^3-36a^2+54a-27(2a-3 \\
 & 8a^3 \\
\hline
3\times(2a)^2=12a^2 & -36a^2+54a-27 \\
3(2a)(-3)=-18a & \\
(-3)^2=9 & \\
\hline
12a^2-18a+9 & -36a^2+54a-27 \\
\hline
\end{array}$$

Hence the cube root is $2a-3$.

Ex. 2. Extract the cube root of:—

$$8x^6+64-144x-36x^5+102x^4+204x^2-171x^3.$$

Arrange the expression according to the descending powers of x.

$$8x^6-36x^5+102x^4-171x^3+204x^2-144x+64\,(2x^2-3x+4.$$
$$8x^6$$

$$\begin{array}{l|l}
3(2x^2)^2=12x^4 & -36x^5+102x^4-171x^3+204x^2-144x+64 \\
3(2x^2)(-3x)=-18x^3 & \\
(-3x)^2=9x^2 & \\
\hline
12x^4-18x^3+9x^2 & -36x^5+54x^4-27x^3 \\
\hline
3(2x^2-3x)^2=12x^4-36x^3+27x^2 & 48x^4-144x^3+204x^2-144x+64 \\
3(2x^2-3x)\times 4=24x^2-36x & \\
4^2=16 & \\
\hline
12x^4-36x^3+51x^2-36x+16 & 48x^4-144x^3+204x^2-144x+64
\end{array}$$

Hence the cube root is $2x^2-3x+4$.

116. *When* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *more may be obtained by division only supposing* 2n + 2 *to be the whole number of figures in the root.*

Let N represent the number whose cube root is required; a the part of the root already found; and x the part to be found; then $\sqrt[3]{N}=a+x$. $\therefore N=a^3+x^3+3a^2x+3ax^2$

$$\because N-a^3=3a^2x+3ax^2+x^3 \quad \therefore \frac{N-a^3}{3a^2}=x+\frac{x^2}{a}+\frac{x^3}{3a^2}.$$

If we shew that $\dfrac{x^2}{a}+\dfrac{x^3}{3a^2}$ is a *proper fraction*, then $x=$ the quotient of $N-a^3$ divided by $3a^2$.

Since x contains n digits, x is less than 10^n. $\therefore$ x^2 is *less* than 10^{2n}. Since a contains $2n+2$ digits, a is *not less* than 10^{2n+1}. $\therefore \dfrac{x^2}{a}$ is *less* than $\dfrac{10^{2n}}{10^{2n+1}}$, i.e., *less than* $\dfrac{1}{10}$.

Since x is less than 10^n, x^3 is *less* than 10^{3n}.

Since a is *not less* than 10^{2n+1}, $3a^2$ is *not less* than $3\times 10^{4n+2}$.

$\therefore \dfrac{x^3}{3a^2}$ is less than $\dfrac{10^{3n}}{3\times 10^{4n+2}}$, i.e., less than $\dfrac{1}{3\times 10^{n+2}}$.

Hence $\frac{x^2}{a}+\frac{x^3}{3a^2}$ is less than $\frac{1}{10}+\frac{1}{3\times 10^{n+4}}$ $\therefore$ it is a *proper fraction.*

EXERCISE 48.

Extract the cube root of :—

1. $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$.
2. $a^3+\frac{1}{a^3}+3\left(a^2+\frac{1}{a^2}\right)+6\left(a+\frac{1}{a}\right)+7$.
3. $\frac{x^6}{64}-\frac{3x^5}{16}+\frac{15x^4}{16}-\frac{5x^3}{2}+\frac{15x^2}{4}-3x+1$.
4. $x^3(x^3+11)-3x^2(x^3-2)-3x(x^3+4)-8$.
5. $1-9x(1+11x^2)+39x^2+156x^4-144x^5+64x^6$.
6. $a^3-\frac{1}{a^3}+3\left(a^2+\frac{1}{a^2}\right)-5$
7. $a^3+\frac{8}{a^3}-12a-\frac{48}{a^2}+54a+\frac{108}{a}-112$.
8. $a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20$.
9. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
10. $x^6+x^3+1+3x(x+1)^2(x^2+1)$.
11. $a^3-1+\frac{1}{a^3}-3\left(a+\frac{1}{a}\right)\left(a+\frac{1}{a}-2\right)$.
12. $x^3-y^3-z^3+3(x-y)(z-x)(y+z)$.

117. The fourth root of an expression is the square root of the square root of the expression. Thus $\sqrt[4]{16}=\sqrt{(\sqrt{16})}=\sqrt{4}=2$.

The sixth root of an expression is the square root of the cube root of the expression; or the cube root of the square root of the expression Thus $\sqrt[6]{64}=\sqrt{(\sqrt[3]{64})}=\sqrt{4}=2$; or $=\sqrt[3]{(\sqrt{64})}=\sqrt[3]{8}=2$.

118. Miscellaneous Examples.

Ex. 1. Extract the cube root of $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$, by *the Method of Indeterminate Co-efficients.*

If the given expression be a perfect cube, it must be the cube of a trinomial of the form x^2+px+1.

$$\therefore x^6+6x^5+15x^4+20x^3+15x^2+6x+1=(x^2+px+1)^3$$
$$=x^6+3px^5+(\text{terms containing } x^4,\ x^3,\ x^2 \text{ and } x)+1.$$

Since this is true for all values of x, the co-efficients of the like powers of x are equal. $\therefore 6=3p \quad \therefore p=2$. This value must be verified by equating the co-efficients of the other powers of x.

Hence the cube root of the given expression is x^2+2x+1.

Ex. 2. What value of x will make $x^6+9x^5+21x^4-9x^3-42x^2+33x+1$ a *perfect* cube?

Rule.—"Extract the cube root by the ordinary method and put the remainder $=0$."

$$
\begin{array}{r|l}
 & x^6+9x^5+21x^4-9x^3-42x^2+33x+1\,(x^2+3x-2 \\
 & x^6 \\
\hline
3(x^2)^2=3x^4 & 9x^5+21x^4-9x^3-42x^2+33x+1 \\
3(x^2)(3x)=9x^3 & \\
(3x)^2=9x^2 & \\
\hline
3x^4+9x^3+9x^2 & 9x^5+27x^4+27x^3 \\
\hline
3(x^2+3x)^2=3x^4+18x^3+27x^2 & -6x^4-36x^3-42x^2+33x+1 \\
3(x^2+3x)(-2)=-6x^2-18x & \\
(-2)^2=\qquad\qquad 4 & \\
\hline
3x^4+18x^3+21x^2-18x+4 & -6x^4-36x^3-42x^2+36x-8 \\
\hline
 & -3x+9 \ \textit{Remainder.}
\end{array}
$$

$\therefore -3x+9=0. \quad \therefore 3x=9. \quad \therefore x=3.$

Ex. 3. Find the relation between a, b, c and d, when ax^3+bx^2+cx+d is a complete cube.

Let $ax^3+bx^2+cx+d=(px+q)^3=p^3x^3+3p^2qx^2+3pq^2x+q^3$.

The co-efficients of the *like* powers of x are equal.

$\therefore a=p^3$; $b=3p^2q$; $c=3pq^2$ and $d=q^3$.

$\therefore \dfrac{a}{d}=\dfrac{p^3}{q^3}=\left(\dfrac{p}{q}\right)^3$; $\dfrac{b}{c}=\dfrac{3p^2q}{3pq^2}=\dfrac{p}{q}. \quad \therefore \dfrac{b^3}{c^3}=\left(\dfrac{p}{q}\right)^3.$

$\therefore \dfrac{a}{d}=\dfrac{b^3}{c^3} \quad \therefore ac^3=b^3d$ . . . . (1).

Again, $ad=p^3q^3$ and $bc=9p^3q^3$. $\therefore 9ad=bc$ . . . . . (2).

Ex. 4. The product of any three consecutive numbers together with the middle number is the cube of the middle number.

i.e., $(a+1)(a+2)(a+3)+a+2=(a+2)^3$.

$$(a+1)(a+2)(a+3)+a+2=(a+2)\{(a+1)(a+3)+1\}$$
$$=(a+2)(a+2)^2=(a+2)^3.$$

Ex 5. The product of any three numbers which increase or decrease by a common difference together with the product of the square of that difference into the middle number the cube of the middle number.

$$(a+b)(a+3b)(a+5b)+(2b)^2(a+3b)=(a+3b)^3.$$

Left side expression $=(a+3b)\{(a+b)(a+5b)+(2b)^2\}$
$=(a+3b)\{a^2+6ab+5b^2+4b^2\}=(a+3b)(a+3b)^2$
$=(a+3b)^3$.

EXERCISE 49.

Extract the cube root of each of the following expressions, by the *Method of Indeterminate Co-efficients.*

1. $x^6-9x^5+33x^4-63x^3+66x^2-36x+8$.
2. $8x^6+48cx^5+60c^2x^4-80c^3x^3-90c^4x^2+108c^5x-27c^6$.
3. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
4. $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.
5. $x^6+15x^2+\frac{15}{x^2}+20+\frac{6}{x^4}+\frac{1}{x^6}+6x^4$.
6. $27x^6-6x-54x^5-44x^3+63x^4+21x^2+1$.

What value of x will make each of the following a *perfect cube* ?—

7. $8x^6-36x^5+66x^4-63x^3+33x^2-11x+6$.
8. $1+6x+21x^2+44x^3+63x^4+58x^5+23x^6$.
9. $x^3-\frac{1}{x^3}-4x^2-\frac{3}{x^2}+6$.
10. $x^6-6x^5+15x^4-20x^3+15x^2-7x+8$.
11. $8x^6+12x^5-6x^4-11x^3+3x^2+5x-27$.

12. Find the *fourth* root of $1-2x+3\frac{x^2}{2}-\frac{x^3}{2}+\frac{x^4}{16}$.

13. Find the *fourth* root of $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$.

14. Find the *sixth* root of $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

15. Find the conditions that $x^3+3ax^2+3bx+c$ should be a *perfect* cube.

16. Find the relation between p, q and r, in order that x^3+px^2+qx+r shall be a *perfect* cube.

17. Find the conditions that $ax^4+bx^3+cx^2+dx+e$ should be a *complete fourth power.*

18. Find the cube root of $1-x$ to 4 terms.

Shew that each of the following is a complete cube :—

19. $\left(\frac{a+b}{a-b}\right)^3+\left(\frac{a-b}{a+b}\right)^3+6\frac{a^2+b^2}{a^2-b^2}$.

20. $(x-b)(x-4b)(x-7b)+9b^2(x-4b)$.

21. $(x+a-b)(x+2a-3b)(x+3a-5b)+(a-2b)^2(x+2a-3b)$.

22. Find the relations between the co-efficients when $x^5+px^4+qx^3+rx^2+sx+t$ is a *complete fifth power.*

CHAPTER XVIII.

INDICES.

119. Definition of a^m. The product of m factors each equal to a is denoted by a^m.

The Index Law is $a^m \times a^n = a^{m+n}$. We have already proved this in Art. 35 when m and n are *positive* integers. We shall assume that the law is true for *all* values of m and n and proceed to find the meanings of a^0, a^{-n} and $a^{\frac{m}{n}}$.

120. (i) *To find the meaning of a^0.*

Now $a^m \times a^n = a^{m+n}$. Let $m=0$; then $a^0 \times a^n = a^{0+n} = a^n$.

$\therefore\ a^0 = \dfrac{a^n}{a^n} = 1.$ Thus any quantity raised to the power *zero* is equal to 1.

(ii) *To find the meaning of a^{-n}.*

Now $a^m \times a^n = a^{m+n}$. Let $m=-n$; then $a^{-n} + a^n = a^{-n+n}$ $= a^0 = 1$.

$$\therefore a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}.$$

Thus, $a^{-2} = \dfrac{1}{a^2}$; $a^{-\frac{1}{2}} = \dfrac{1}{a^{\frac{1}{2}}}$; $a^{m-n} = \dfrac{1}{a^{n-m}}$.

a^{-n} is the *reciprocal* of a^n.

(iii) *To find the meaning of $a^{\frac{m}{n}}$, when* m *and* n *are positive integers.*

$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots$ to n factors $= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n} \ldots \text{ to } n \text{ terms}}$

$$= a^{\frac{m}{n}\times n} = a^m.$$

Hence $a^{\frac{m}{n}}$ is the n^{th} root of a^m ; i.e , $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

In a fractional index, the numerator denotes a power and the denominator a root.

Thus, $a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a}$; $a^{\frac{2}{3}} = \sqrt[3]{a^2}$; $a^{\frac{1}{4}} = \sqrt[4]{a}$.

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(a^p\right)^{\frac{1}{q}}.$$

121. In like manner we can prove that—

(i) $(a^m)^n = a^{mn}$; (ii) $a^n b^n = (ab)^n$

(iii) $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$ for *all* values of m and n.

Ex. 1. Express, with fractional indices, $\sqrt[6]{a^7 b^{10} c^9 d^{-15}}$

$$\sqrt[6]{a^7 b^{10} c^9 d^{-15}} = (a^7 b^{10} c^9 d^{-15})^{\frac{1}{6}} = a^{\frac{7}{6}} b^{\frac{10}{6}} c^{\frac{9}{6}} d^{-\frac{15}{6}} = a^{\frac{7}{6}} b^{\frac{5}{3}} c^{\frac{3}{2}} d^{-\frac{5}{2}}.$$

Ex. 2. Simplify $x^{2a-b-c} \,.\, x^{2b-c-a} \,.\, x^{2c-a-b}$.

The expression $= x^{(2a-b-c+2b-c-a+2c-a-b)} = x^0 = 1$.

Ex. 3. Simplify $\{(a+b)^{m-n}(a+b)^{m+n}\}^{\frac{1}{4mn}}$

The expression $= \{(a+b)^{m-n+m+n}\}^{\frac{1}{4mn}} = \{(a+b)^{2m}\}^{\frac{1}{4mn}}$

$= (a+b)^{\frac{2m}{4mn}} = (a+b)^{\frac{1}{2n}}$.

Ex. 4. Multiply $x^{-\frac{1}{2}} - x^{-\frac{1}{4}} y^{-\frac{1}{4}} + y^{-\frac{1}{2}}$ by $x^{-\frac{1}{4}} + y^{-\frac{1}{4}}$.

Putting a for $x^{-\frac{1}{4}}$, b for $y^{-\frac{1}{4}}$, multiplicand $= a^2 - ab + b^2$ and multiplier $= a + b$. Hence the product $= (a^2 - ab + b^2)(a + b)$ $= a^3 + b^3 = \left(x^{-\frac{1}{4}}\right)^3 + \left(y^{-\frac{1}{4}}\right)^3 = x^{-\frac{3}{4}} + y^{-\frac{3}{4}}$.

Ex. 5. Divide $a - 2a^{\frac{1}{2}} + 1$ by $a^{\frac{1}{3}} - 2a^{\frac{1}{6}} + 1$.

Putting x for $a^{\frac{1}{6}}$, the dividend $= x^6 - 2x^3 + 1$; and the divisor $= x^2 - 2x + 1$. ∵ $a = \left(a^{\frac{1}{6}}\right)^6$; $a^{\frac{1}{2}} = \left(a^{\frac{1}{6}}\right)^3$; $a^{\frac{1}{3}} = \left(a^{\frac{1}{6}}\right)^2$.

Hence $$\frac{a - 2a^{\frac{1}{2}} + 1}{a^{\frac{1}{3}} - 2a^{\frac{1}{6}} + 1} = \frac{x^6 - 2x^3 + 1}{x^2 - 2x + 1} = \frac{(x^3 - 1)^2}{(x - 1)^2} = \left(\frac{x^3 - 1}{x - 1}\right)^2$$

$$=(x^2+x+1)^2=x^4+2x^3+3x^2+2x+1$$
$$=\left(a^{\frac{1}{6}}\right)^4+2\left(a^{\frac{1}{6}}\right)^3+3\left(a^{\frac{1}{6}}\right)^2+2a^{\frac{1}{6}}+1$$
$$=a^{\frac{2}{3}}+2a^{\frac{1}{2}}+3a^{\frac{1}{3}}+2a^{\frac{1}{6}}+1.$$

Ex. 6. Extract the square root of $a+b^{\frac{2}{3}}-2a^{\frac{1}{2}}b^{\frac{1}{3}}+c^{\frac{1}{2}}+2a^{\frac{1}{2}}c^{\frac{1}{4}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}$.

Arrange the expression according to powers of a.

$$\begin{array}{l|l|l}
a^{\frac{1}{2}} & a-2a^{\frac{1}{2}}b^{\frac{1}{3}}+2a^{\frac{1}{2}}c^{\frac{1}{4}}+b^{\frac{2}{3}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}} & a^{\frac{1}{2}}-b^{\frac{1}{3}}+c^{\frac{1}{4}} \\
 & \underline{a} & \\
2a^{\frac{1}{2}}-b^{\frac{1}{3}} & -2a^{\frac{1}{2}}b^{\frac{1}{3}}+2a^{\frac{1}{2}}c^{\frac{1}{4}}+b^{\frac{2}{3}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}} & \\
 & \underline{-2a^{\frac{1}{2}}b^{\frac{1}{3}}\qquad\qquad +b^{\frac{2}{3}}} & \\
2a^{\frac{1}{2}}-2b^{\frac{1}{3}}+c^{\frac{1}{4}} & 2a^{\frac{1}{2}}c^{\frac{1}{4}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}} & \\
 & \underline{2a^{\frac{1}{2}}c^{\frac{1}{4}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}}} &
\end{array}$$

Hence, the square root is $a^{\frac{1}{2}}-b^{\frac{1}{3}}+c^{\frac{1}{4}}$.

Ex. 7. Simplify $\dfrac{1}{1+x^{m-n}+x^{m-p}}+\dfrac{1}{1+x^{n-m}+x^{n-p}}$

$$+\frac{1}{1+x^{p-m}+x^{p-n}}.$$

The first term $=\dfrac{x^{-m}}{x^{-m}(1+x^{m-n}+x^{m-p})}=\dfrac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}$.

The second term $=\dfrac{x^{-n}}{x^{-n}(1+x^{n-m}+x^{n-p})}=\dfrac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}}$.

The third term $=\dfrac{x^{-p}}{x^{-p}(1+x^{p-m}+x^{p-n})}=\dfrac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}$.

Hence, the given expression

$$=\frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}+\frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}}+\frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}$$

$$=\frac{x^{-m}+x^{-n}+x^{-p}}{x^{-m}+x^{-n}+x^{-p}}=1.$$

Ex. 8. If $a^b=b^a$, shew that $\left(\frac{a}{b}\right)^{\frac{a}{b}}=a^{\frac{a}{b}-1}$; and if $a=2b$, shew that $b=2$.

Since $a^b=b^a$, $\therefore a=b^{\frac{a}{b}}$. Hence $\left(\frac{a}{b}\right)^{\frac{a}{b}}=\frac{a^{\frac{a}{b}}}{b^{\frac{a}{b}}}=\frac{a^{\frac{a}{b}}}{a}=a^{\frac{a}{b}-1}$.

Since $a^b=b^a$ and $a=2b$, $\therefore (2b)^b=b^{2b}=(b^2)^b$ $\therefore 2b=b^2$ $\therefore b=2$.

Ex. 9. If $x=(a+\sqrt{a^2+b^3})^{\frac{1}{3}}+(a-\sqrt{a^2+b^3})^{\frac{1}{3}}$, shew that $x^3+3bx-2a=0$.

Putting p for $a+\sqrt{a^2+b^3}$ and q for $a-\sqrt{a^2+b^3}$, we have $x=p^{\frac{1}{3}}+q^{\frac{1}{3}}$ $\therefore x^3=(p^{\frac{1}{3}}+q^{\frac{1}{3}})^3$ $\therefore x^3=p+q+3p^{\frac{1}{3}}q^{\frac{1}{3}}(p^{\frac{1}{3}}+q^{\frac{1}{3}})=p+q+3p^{\frac{1}{3}}q^{\frac{1}{3}}(x)$. But $p+q=a+\sqrt{a^2+b^3}+a-\sqrt{a^2+b^3}=2a$ and $pq=(a+\sqrt{a^2+b^3})(a-\sqrt{a^2+b^3})=a^2-(a^2+b^3)=-b^3$ $\therefore p^{\frac{1}{3}}q^{\frac{1}{3}}=(-b^3)^{\frac{1}{3}}=-b$ $\therefore x^3=2a+3(-b)(x)=2a-3bx$ $\therefore x^3+3bx-2a=0$.

EXERCISE 50.

1. Multiply $a^{-2}-1+a^2$ by $a^{-1}+a$.
2. Multiply $a^{\frac{3}{4}}+a^{\frac{1}{2}}+a^{\frac{1}{4}}+1$ by $a^{\frac{1}{4}}-1$.
3. Multiply $x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{\frac{1}{4}}y$ by $x^{\frac{1}{4}}-y^{\frac{1}{2}}$.
4. Simplify $\sqrt[a-b]{(\sqrt[a-c]{x^a})^a}\times\sqrt[b-c]{(\sqrt[b-a]{x^b})^b}\times\sqrt[c-a]{(\sqrt[c-b]{x^c})^c}$.
5. Divide $a-b^{\frac{3}{2}}+c^{\frac{3}{4}}+3a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}}$ by $a^{\frac{1}{3}}-b^{\frac{1}{2}}+c^{\frac{1}{4}}$.
6. Divide $x^{\frac{3p}{2}}-y^{\frac{3p}{2}}$ by $x^{\frac{p}{2}}-y^{\frac{p}{2}}$.
7. Simplify $(\sqrt[bc]{x})^{b-c}\times(\sqrt[ca]{x})^{c-a}\times(\sqrt[ab]{x})^{a-b}$.
8. Multiply $a+b^{\frac{2}{3}}+c^{\frac{1}{2}}-a^{\frac{1}{2}}b^{\frac{1}{3}}-a^{\frac{1}{2}}c^{\frac{1}{4}}-b^{\frac{1}{3}}c^{\frac{1}{4}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{3}}+c^{\frac{1}{4}}$.

9. Simplify $\dfrac{a^{\frac{1}{2}}}{(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}-c^{\frac{1}{2}})}+\dfrac{b^{\frac{1}{2}}}{(b^{\frac{1}{2}}-a^{\frac{1}{2}})\ (b^{\frac{1}{2}}-c^{\frac{1}{2}})}$

$$+\frac{c^{\frac{1}{2}}}{(c^{\frac{1}{2}}-a^{\frac{1}{2}})\ (c^{\frac{1}{2}}-b^{\frac{1}{2}})}.$$

10. Resolve into factors $x^{-2}-7x^{-1}+12$; $x^{\frac{3}{4}}-y^{\frac{3}{4}}$; a^4+b^4; $x^3+x^{-3}-2$; and $(a^{-1}+a^{-2}+a^{-3})(a^{-3}+a^{-5}+a^{-4})-a^{-6}$.

11. Find the H.C.F. and L.C.M. of—

$x(x+\sqrt{xy})-y(y+\sqrt{xy})$ and $x^2+y^2+3(x+y)\sqrt{xy}+4xy$.

12. Express $\left\{\left(\dfrac{a^{-m}}{b^{-n}}\right)^{-\frac{1}{p}}\times\left(\dfrac{b^{-\frac{m}{p}}}{a^{-\frac{n}{p}}}\right)\right\}^{\frac{q}{m+n}}$ in the simplest form and what must be the relation between p and q so that the expression may be equal to $\dfrac{a}{b}$.

13. Simplify $\dfrac{2^{2n+1}+2^{n+2}+2}{2^{n+1}+2}+\dfrac{2^{2n}-1}{2^n+1}+\dfrac{2^{3n}+1}{2^n+1}$.

14. Extract the square root of $y^2x^{-1}+\frac{1}{4}x^2y^{-1}+\dfrac{2y^{\frac{3}{2}}-x^{\frac{3}{2}}}{x^{\frac{1}{4}}y^{\frac{1}{4}}}$.

15. Extract the square root of $4x-12x^{\frac{1}{2}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}+16x^{\frac{1}{2}}z^{\frac{1}{4}}$ $-24y^{\frac{1}{2}}z^{\frac{1}{4}}+16z^{\frac{1}{2}}$.

16. Find the square of $x^{\frac{1}{3}}-2x^{\frac{1}{4}}+x^{\frac{1}{2}}$.

17. Simplify $\dfrac{a^{\frac{3}{2}}-ax^{\frac{1}{2}}+a^{\frac{1}{2}}x-x^{\frac{3}{2}}}{a^{\frac{5}{2}}-a^2x^{\frac{1}{2}}+3a^{\frac{3}{2}}x-3ax^{\frac{3}{2}}+a^{\frac{1}{2}}x^2-x^{\frac{5}{2}}}$.

18. Simplify $\dfrac{1+4x^{\frac{1}{2}}-2x-12x^{\frac{3}{2}}+9x^2}{1-4x^{\frac{1}{2}}+6x-4x^{\frac{3}{2}}+x^2}$.

CHAPTER XIX.

SURDS.

122. Definitions.

A Surd or an Irrational Qantity. When any root of a quantity cannot be exactly extracted, the quantity which represents the root is called *a surd*.

Thus $\sqrt{2}, \sqrt{6}, \sqrt[3]{9}, \sqrt{2}+\sqrt[3]{3}, 3^{\frac{1}{4}}-5^{\frac{1}{2}}$ are surds.

Such expressions as $\sqrt{4}, \sqrt[4]{\frac{16}{81}}, \sqrt{a^2-2ab+b^2}$ are not surds though they are written in surd form.

Though such expressions as $\sqrt{a}$, $ab^{\frac{1}{2}}, a^{\frac{2}{3}}, (a^2+b^2)^{\frac{1}{3}}$ are surds, yet their exact values may be found in certain cases by giving special values to the letters a and b. For instance if $a=4$, $\sqrt{a}=\sqrt{4}=2$ and is therefore not really a surd.

Surd-factor is the quantity beneath the radical or index sign : thus 3 is the surd-factor of $4\sqrt{3}$ and $6\sqrt{3}$.

A Complete Surd is one which has no rational co-efficient except unity. Thus $\sqrt{2}$ and $(x+y)^{\frac{1}{2}}$ are complete surds.

A Mixed Surd is one which has a rational factor different from unity. Thus $3\sqrt{2}$, $x\sqrt[3]{3}$, $4x^{\frac{1}{4}}$ are mixed surds.

Simple and Compound Surds. Surds are *simple* and *compound* according as they consist of one, or more than one term.

Similar Surds are those which have, or can be transformed so as to have, the same irrational part. Thus $\sqrt{45}$ and $\sqrt{80}$ are similar surds for they are respectively equivalent to $3\sqrt{5}$ and $4\sqrt{5}$.

Surds of the same order. Surds are said to be *of the same order* when they have all got the same root symbol.

A surd is said to be of the *second*, *third*, or n^{th} order, according as the denominator of the surd index is 2, 3, or n. Thus $\sqrt{5}$, $\sqrt{a^3}$ and $\sqrt{(a+b)}$ are surds of the *second* order.

A Quadratic Surd is a surd of the second order.

Ex. 1. Express $3\sqrt{5}$ in the form of a *complete* surd.

$$3\sqrt{5}=(3^2)^{\frac{1}{2}}\times 5^{\frac{1}{2}}=(3^2\times 5)^{\frac{1}{2}}=(45)^{\frac{1}{2}}=\sqrt{45}.$$

Ex. 2. Express $\sqrt{80}$ as the product of a rational quantity and a surd.

$\sqrt{80}=\sqrt{16\times5}=(16\times5)^{\frac{1}{2}}=(4^2\times5)^{\frac{1}{2}}=(4^2)^{\frac{1}{2}}\times5^{\frac{1}{2}}=4\times5^{\frac{1}{2}}$ $=4\sqrt{5}$.

Ex. 3. Find the sum of $\sqrt{98}\pm\sqrt{50}$ and of $\sqrt[3]{128}-\sqrt[3]{54}$.

$\sqrt{98}\pm\sqrt{50}=\sqrt{49\times2}\pm\sqrt{25\times2}=7\sqrt{2}\pm5\sqrt{2}=12\sqrt{2}$ or $2\sqrt{2}$; $\sqrt[3]{128}-\sqrt[3]{54}=\sqrt[3]{64\times2}-\sqrt[3]{27\times2}=\sqrt[3]{4^3\times2}-\sqrt[3]{3^3\times2}$ $=4\sqrt[3]{2}-3\sqrt[3]{2}=\sqrt[3]{2}$.

Ex. 4. Reduce $\sqrt{3}$ and $\sqrt[3]{2}$ to surds of the *same* order.

$\sqrt{3}$ is of the *second* order; $\sqrt[3]{2}$ is of the *third* order.

The L.C.M. of 2 and 3 is 6; so we can reduce them to surds of the *sixth* order, thus: $\sqrt{3}=3^{\frac{1}{2}}=3^{\frac{3}{6}}=\sqrt[6]{3^3}=\sqrt[6]{27}$; and $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{2}{6}}=\sqrt[6]{2^2}=\sqrt[6]{4}$.

Ex. 5. Which is the greater, $\sqrt{2}$ or $\sqrt[3]{3}$?

Reduce them to surds of the same order. L.C.M. of 2 and 3 is 6.

$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{3}{6}}=\sqrt[6]{2^3}=\sqrt[6]{8}$; $\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{2}{6}}=\sqrt[6]{3^2}=\sqrt[6]{9}$.

$\sqrt{2}$ and $\sqrt[3]{3}$ are respectively equivalent to $\sqrt[6]{8}$ and $\sqrt[6]{9}$ and as 9 is greater than 8, therefore $\sqrt[3]{3}>\sqrt{2}$.

Ex. 6. Multiply $\sqrt[4]{3}$ by $\sqrt[3]{9}$; $\sqrt[3]{3}$ by $\sqrt[4]{3}$.

$$\sqrt[4]{3}\times\sqrt[3]{9}=3^{\frac{1}{4}}\times9^{\frac{1}{3}}=3^{\frac{3}{12}}\times9^{\frac{4}{12}}=(3^3)^{\frac{1}{12}}\times(9^4)^{\frac{1}{12}}$$

$$=(3^3\times9^4)^{\frac{1}{12}}=(27\times81\times81)^{\frac{1}{12}}=(177147)^{\frac{1}{12}}=\sqrt[12]{177147}.$$

$$\sqrt[3]{3}\times\sqrt[4]{3}=3^{\frac{1}{3}}\times3^{\frac{1}{4}}=3^{\frac{1}{3}+\frac{1}{4}}=3^{\frac{7}{12}}=\sqrt[12]{3^7}=\sqrt[12]{2187}.$$

Ex. 7. Divide $\sqrt[4]{5}$ by $\sqrt[3]{10}$ and express $\sqrt{3}\div2\sqrt{5}$ as a fraction with a rational denominator.

$$\sqrt[4]{5}\div\sqrt[3]{10}=5^{\frac{1}{4}}\div10^{\frac{1}{3}}=5^{\frac{3}{12}}\div10^{\frac{4}{12}}=\frac{(5^3)^{\frac{1}{12}}}{(10^4)^{\frac{1}{12}}}=\left(\frac{5^3}{10^4}\right)^{\frac{1}{12}}$$

$$=\left(\frac{1}{80}\right)^{\frac{1}{12}}=\frac{1}{\sqrt[12]{80}}.$$

$$\sqrt{3}\div2\sqrt{5}=\frac{\sqrt{3}}{2\sqrt{5}}=\frac{\sqrt{3}\times\sqrt{5}}{2\sqrt{5}\times\sqrt{5}}=\frac{\sqrt{15}}{2\times5}=\frac{\sqrt{15}}{10}.$$

Ex. 8. Multiply $2\sqrt{a}+3\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.

$(2\sqrt{a}+3\sqrt{b})(\sqrt{a}-\sqrt{b}) = 2\sqrt{a}\sqrt{a}-2\sqrt{a}\sqrt{b}+3\sqrt{b}\sqrt{a}-3\sqrt{b}\sqrt{b}$

$=2a-2\sqrt{ab}+3\sqrt{ab}-3b=2a+\sqrt{ab}-3b.$

Ex. 9. Multiply $5\sqrt{7}+4\sqrt{3}$ by $5\sqrt{7}-4\sqrt{3}$.

$(5\sqrt{7}+4\sqrt{3})(5\sqrt{7}-4\sqrt{3})=(5\sqrt{7})^2-(4\sqrt{3})^2=5^2\times 7-4^2\times 3=175-48=127.$

Ex. 10. Find the square of $\sqrt{a+x}+\sqrt{a-x}$.

$(\sqrt{a+x}+\sqrt{a-x})^2=(\sqrt{a+x})^2+(\sqrt{a-x})^2+2\sqrt{a+x}\sqrt{a-x}$
$=a+x+a-x+2\sqrt{a^2-x^2}=2a+2\sqrt{a^2-x^2}.$

EXERCISE 51.

Simplify :—

1. $\sqrt[3]{625}-\sqrt[3]{135}+\sqrt[3]{40}$. 2. $12\sqrt{2}+37\sqrt{8}-9\sqrt{50}$.

3. $\sqrt[3]{-8x^3}-\sqrt[4]{16x^4}$.

4. $\sqrt[4]{32b^4a}+2\sqrt[4]{512b^4a}-6b\sqrt[4]{1250a}$.

5. $\sqrt[3]{3}+\dfrac{2}{\sqrt[3]{9}}-\sqrt[3]{24}$. 6. $\sqrt[3]{-27a^3b^3}+\sqrt{4a^2b^2}-\sqrt[4]{16a^4b^4}$.

7. Arrange in order of magnitude : $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[6]{6}$.

8. Which is the greater : $\sqrt[3]{3}$ or $\sqrt[4]{4}$; $\sqrt[3]{6}$ or $\sqrt[4]{10}$.

9. Reduce $\sqrt{a^2b}$, $\sqrt{x^3y^3}$, $\sqrt[3]{x^4y^3}$, $\sqrt{x^3+x^2-x-1}$ and $\sqrt[3]{\dfrac{x-3}{4x^2-24x+36}}$.

10. Reduce to surds of the *same* order $\sqrt{x}$ and $\sqrt[4]{y}$; $\sqrt[3]{4}$ and $\sqrt[4]{3}$.

11. Compare $3\sqrt{7}$, $4\sqrt{2}$ and $4\sqrt[3]{3}$.

12. Multiply $2\sqrt{3}$ by $3\sqrt{27}$; $\sqrt[3]{4}$ by $\sqrt{8}$; $8\sqrt{12}$ by $3\sqrt{24}$; $\sqrt[3]{6ab}$ by $\sqrt[3]{9a^2b^3}$.

13. Multiply together $4\sqrt{2\frac{1}{2}}$, $\sqrt{1\frac{1}{2}}$ and $\sqrt{5}$; $12\sqrt[3]{2}$, $2\sqrt[3]{18}$ and $\dfrac{1}{\sqrt[3]{12}}$; $(\sqrt{2}+\sqrt{3})$ and $\sqrt{3}-\sqrt{2}$.

14. Reduce $(2\sqrt{3}+3\sqrt{2})(2\sqrt{2}+\sqrt{3})\div 2\sqrt{3}$.

15. Reduce $\sqrt{a+\sqrt{a^2-b^2}}\times\sqrt{a-\sqrt{a^2-b^2}}$.

16. Find the square of $\sqrt{x+2a}-\sqrt{x-2a}$; $a\sqrt{5}+b\sqrt{5}$; $\sqrt{x^2+2y^2}-\sqrt{x^2-2y^2}$ and $2\sqrt{a+b}+3\sqrt{a-b}$.

17. Multiply $\sqrt{a}+\sqrt{b}+\sqrt{c}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

18. Find the continued product of $\sqrt{a}+\sqrt{b}-\sqrt{c}$, $\sqrt{a}+\sqrt{c}-\sqrt{b}$, $\sqrt{c}+\sqrt{b}-\sqrt{a}$ and $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

19. Shew that the product of $\sqrt[3]{a^2+\sqrt{a^2-b^3}}$ and $\sqrt[3]{a^2-\sqrt{a^2-b^3}}$ is rational.

20. Find the continued product of $\sqrt{3}+\sqrt{2}+2$, $\sqrt{2}+2-\sqrt{3}$, $2+\sqrt{3}-\sqrt{2}$ and $\sqrt{3}+\sqrt{2}-2$.

123. Rationalising Factor. A rationalising factor of a surd is one which when multiplied by the surd gives a rational product.

Since $\sqrt{a}\times\sqrt{a}=a$, therefore $\sqrt{a}$ is a rationalising factor of $\sqrt{a}$.

Since $a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a$, therefore $a^{\frac{1}{3}}$ is a rationalising factor of $a^{\frac{2}{3}}$, and *vice versâ*.

If the sum of the surd-indices of two simple similar surds is an integer, either of them is a rationalising factor of the other.

Since $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$, therefore each of $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ is a rationalising factor of the other.

Conjugate or Complementary Surds. Two binomial quadratic surds *whose* product is rational are called *conjugate* surds.

To rationalise a binomial quadratic surd, we must multiply it by the complementary surd

Ex. 1. Find a rationalising factor of $3\sqrt{5}+2\sqrt{7}$.

The conjugate surd is $3\sqrt{5}-2\sqrt{7}$; their product is $(3\sqrt{5}+2\sqrt{7})(3\sqrt{5}-2\sqrt{7})=(3\sqrt{5})^2-(2\sqrt{7})^2=45-28=17$. $\therefore$ the rationalising factor is $3\sqrt{5}-2\sqrt{7}$.

Ex. 2. Rationalise the denominator of $\dfrac{1+\sqrt{3}}{4-2\sqrt{3}}$ and find its value when $\sqrt{3}=1.732$.

$$\frac{1+\sqrt{3}}{4-2\sqrt{3}}=\frac{(1+\sqrt{3})(4+2\sqrt{3})}{(4-2\sqrt{3})(4+2\sqrt{3})}=\frac{4+4\sqrt{3}+2\sqrt{3}+6}{16-12}=\frac{10+6\sqrt{3}}{4}$$

$$=\frac{5+3\sqrt{3}}{2}=\frac{5+3\times1\cdot732}{2}=\frac{10\cdot196}{2}=5\cdot098.$$

Note.—In similar cases, we should first rationalise the denominator before substituting the value.

Ex. 3. Simplify $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}$.

The expression $= \dfrac{\sqrt{3}(\sqrt{3}+1)(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}$

$$= \frac{\sqrt{3}(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})} = \frac{\sqrt{3}(3\sqrt{5}-6+5-2\sqrt{5})}{\sqrt{5}(\sqrt{5}-1)}.$$

$$= \frac{\sqrt{3}(\sqrt{5}-1)}{\sqrt{5}(\sqrt{5}-1)} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}\times\sqrt{5}}{5} = \frac{\sqrt{15}}{5}.$$

Ex. 4. Find the value of $4x^3+16x^2+10x-1$, when $x=\frac{1}{2}(-1+\sqrt{3})$.

Since $x=\frac{1}{2}(-1+\sqrt{3})$, $2x=-1+\sqrt{3}$. $\therefore\ 2x+1=\sqrt{3}$.
$\therefore\ (2x+1)^2=3$. $\therefore\ 4x^2+4x+1=3$. $\therefore\ 4x^2+4x-2=0$.
$\therefore\ 2x^2+2x-1=0$.

Now $4x^3+16x^2+10x-1=2x(2x^2+2x-1)+6(2x^2+2x-1)+5=5$. $\because\ 2x^2+2x-1=0$.

EXERCISE 52.

Rationalise the denominators of :—

1. $\dfrac{1}{\sqrt{2}-1}$. 2. $\dfrac{4+\sqrt{2}}{3-2\sqrt{2}}$. 3. $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$.

4. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. 5. $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$.

6. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$; $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

7. Find the values of $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$; $\dfrac{4+\sqrt{5}}{5-2\sqrt{5}}$ and $\dfrac{1+4\sqrt{2}}{3\sqrt{2}-4}$

each to 3 places of decimals.

Simplify :—

8. $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}}+\dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$. 9. $\dfrac{2+\sqrt{3}}{4+\sqrt{3}}+\dfrac{4-2\sqrt{3}}{5-2\sqrt{3}}$.

10. $\dfrac{1}{a+\sqrt{a^2-b^2}}+\dfrac{1}{a-\sqrt{a^2-b^2}}$.

11. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$

12. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}.$

13. $\left(\dfrac{\sqrt{1+x}}{\sqrt{1-x}}-\dfrac{\sqrt{1-x}}{\sqrt{1+x}}\right)\div\left(\dfrac{\sqrt{1+x}}{\sqrt{1-x}}+\dfrac{\sqrt{1-x}}{\sqrt{1+x}}\right).$

14. Find the value of $2x^5-3x^4-20x^3+30x^2+2x-2$, whe $x=\sqrt{2}-\sqrt{3}$.

15. If $x+y=\sqrt{m}$, and $x-y=\sqrt{n}$, express x^3+y^3 in term of m and n

124. To find a factor which will rationalise any binomia

First, let the binomial be $a^{\frac{1}{p}}+b^{\frac{1}{q}}$. Put $x=a^{\frac{1}{p}}$, $y=b^{\frac{1}{q}}$; let n be the L.C M. of p and q; then x^n and y^n are both rationa

But $x^n \pm y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ldots\ldots\pm y^{n-1}$ according as n is *odd* or *even*.

$\therefore x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ldots\pm y^{n-1}$ is a factor which wi rationalise $x+y$,

I.E., $a^{\frac{n-1}{p}}-a^{\frac{n-2}{p}}b^{\frac{1}{q}}+\ldots\ldots\pm b^{\frac{n-1}{q}}$ is the rationalising facto of $a^{\frac{1}{p}}+b^{\frac{1}{q}}$

Secondly, let the binomial be $a^{\frac{1}{p}}-b^{\frac{1}{q}}$. Taking x, y and as in the preceding portion. x^n-y^n is a rational quantity.

Now, $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots\ldots+y^{n-1})$.

$\therefore x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots+y^{n-1}$ is a factor which wi rationalise $x-y$,

I.E., $a^{\frac{n-1}{p}}+a^{\frac{n-2}{p}}b^{\frac{1}{q}}+\ldots+b^{\frac{n-1}{q}}$ is a factor which wi rationalise $a^{\frac{1}{p}}-b^{\frac{1}{q}}$.

Ex. Take $a^{\frac{1}{2}}+b^{\frac{1}{3}}$; L.C.M. of 2 and 3=6.

and $$\frac{(a^{\frac{1}{2}})^6-(b^{\frac{1}{3}})^6}{a^{\frac{1}{2}}+b^{\frac{1}{3}}}=a^{\frac{5}{2}}-a^{\frac{4}{2}}b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}-a^{\frac{2}{2}}b^{\frac{3}{3}}+a^{\frac{1}{2}}b^{\frac{4}{3}}-b^{\frac{5}{3}},$$

I.E., $a^{\frac{5}{2}}-a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}-ab+a^{\frac{1}{2}}b^{\frac{4}{3}}-b^{\frac{5}{3}}$ is the rationalising factor of $a^{\frac{1}{2}}+b^{\frac{1}{3}}$ and the rational product is $(a^{\frac{1}{2}})^6-(b^{\frac{1}{3}})^6=a^3-b^2$.

125. Properties of Quadratic surds.

(1) *The square root of a rational quantity cannot be partly rational and partly a quadratic surd.*

If possible, let $\sqrt{x}=a+\sqrt{b}$; then, by squaring, $x=a^2+b+2a\sqrt{b}$. $\therefore \sqrt{b}=\frac{x-a^2-b}{2a}$, a rational quantity, which is contrary to supposition.

(2) *The product of two dissimilar surds cannot be rational.*

Let $\sqrt{x}$ and $\sqrt{y}$ be dissimilar, and, if possible, let $\sqrt{x}\times\sqrt{y}=a$. Squaring, $xy=a^2$. $\therefore y=\frac{a^2}{x}=\frac{a^2}{x^2}x$. $\therefore \sqrt{y}=\left(\frac{a}{x}\right)\sqrt{x}$; this shows that $\sqrt{y}$ and $\sqrt{x}$ are similar surds; which is contrary to supposition.

(3) *A simple quadratic surd cannot be equal to the sum of two dissimilar quadratic surds.*

If possible, let $\sqrt{x}=\sqrt{a}+\sqrt{b}$. Squaring, $x=a+b+2\sqrt{ab}$. $\therefore \sqrt{ab}=\frac{x-a-b}{2}$, a rational quantity; which is impossible by (2). Hence $\sqrt{x}$ is not equal to $\sqrt{a}+\sqrt{b}$.

(4) *If* $x+\sqrt{y}=a+\sqrt{b}$; then $x=a$ *and* $y=b$.

If possible, let $x=a+m$; then $a+m+\sqrt{y}=a+\sqrt{b}$.

$\therefore m+\sqrt{y}=\sqrt{b}$; which is impossible by (1).

Hence $x=a$; and, consequently, $y=b$.

(5) *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$; *then* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Since $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, $\therefore a+\sqrt{b}=x+y+2\sqrt{xy}$.
$\therefore a=x+y$; $\sqrt{b}=2\sqrt{xy}$...by (4). $\therefore a-\sqrt{b}=x+y-2\sqrt{xy}$
$=(\sqrt{x}-\sqrt{y})^2$. $\therefore \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

126. *To find the square root of* $a+\sqrt{b}$.

Let $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, ... (1)

then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$ . . . . (by 5 of Art. 125).

Multiplying, $\sqrt{a^2-b}=x-y$. . . . . . . (2)

Squaring (1), $a+\sqrt{b}=x+y+2\sqrt{xy}$.

(By 4 of Art. 125), $a=x+y$; and $\sqrt{a^2-b}=x-y$.

$\therefore 2x=a+\sqrt{a^2-b}$. $\therefore x=\frac{1}{2}(a+\sqrt{a^2-b})$

and $2y=a-\sqrt{a^2-b}$. $\therefore y=\frac{1}{2}(a-\sqrt{a^2-b})$.

$\therefore \sqrt{a+\sqrt{b}}=\sqrt{\frac{1}{2}(a+\sqrt{a^2-b})}+\sqrt{\frac{1}{2}(a-\sqrt{a^2-b})}$.

Similarly $\sqrt{a-\sqrt{b}}=\sqrt{\frac{1}{2}(a+\sqrt{a^2-b})}-\sqrt{\frac{1}{2}(a-\sqrt{a^2-b})}$.

Note.—Unless a^2-b be a *perfect square*, the values of $\sqrt{x}$ and $\sqrt{y}$ will be complex surds.

Ex. Find the square root of $7-2\sqrt{10}$.

Let $\sqrt{7-2\sqrt{10}}=\sqrt{x}-\sqrt{y}$; then $\sqrt{7+2\sqrt{10}}=\sqrt{x}+\sqrt{y}$.

$\therefore \sqrt{7^2-2^2\times 10}=x-y$. $\therefore x-y=\sqrt{49-40}=\sqrt{9}=3$.

Squaring both sides of $\sqrt{7-2\sqrt{10}}=\sqrt{x}-\sqrt{y}$, we have $7-2\sqrt{10}=x+y-2\sqrt{xy}$. $\therefore x+y=7$ and $x-y=3$. $\therefore x=5$ and $y=2$. $\therefore \sqrt{7-2\sqrt{10}}=\sqrt{5}-\sqrt{2}$.

127. Sometimes we may extract the square root of a quantity of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$, by assuming

$$\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}=\sqrt{x}+\sqrt{y}+\sqrt{z}.$$

$\therefore a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}$.

We may then put $2\sqrt{xy}=\sqrt{b}$, $2\sqrt{yz}=\sqrt{c}$, $2\sqrt{zx}=\sqrt{d}$; *and if the values of x, y and z found from these equations also satisfy the relation* $x+y+z=a$, we shall have the required square root.

Ex. Find the square root of $8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}$.

Assume $\sqrt{(8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10})}=\sqrt{x}+\sqrt{y}+\sqrt{z}$; then $8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}=x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}$.

Put $2\sqrt{xy}=2\sqrt{2}$; $2\sqrt{yz}=2\sqrt{5}$; $2\sqrt{zx}=2\sqrt{10}$.

∴ $\sqrt{xy}=\sqrt{2}$; $\sqrt{yz}=\sqrt{5}$; $\sqrt{zx}=\sqrt{10}$.

$$\therefore \quad \frac{\sqrt{xy}\times\sqrt{yz}}{\sqrt{zx}} \text{ or } \frac{\sqrt{xy^2z}}{\sqrt{zx}} \text{ or } \sqrt{y^2} \text{ or } y=\frac{\sqrt{2}\times\sqrt{5}}{\sqrt{10}}=\frac{\sqrt{10}}{\sqrt{10}}=1.$$

Hence $x=2$ and $z=5$. These values satisfy the relation $x+y+z=8$. Thus the square root required is $\sqrt{2}+\sqrt{1}+\sqrt{5}$, *i.e.*, $1+\sqrt{2}+\sqrt{5}$.

128. If $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, then $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

Since $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, ∴ $a+\sqrt{b}=(x+\sqrt{y})^3=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$.

∴ $x^3+3xy=a$; $\sqrt{b}=3x^2\sqrt{y}+y\sqrt{y}$, ..[By (4) of Art. 125].

Hence $a-\sqrt{b}=x^3+3xy-3x^2\sqrt{y}-y\sqrt{y}=(x-\sqrt{y})^3$.

∴ $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

To find the cube root of $a+\sqrt{b}$.

Let $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$; then $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

By multiplication, $\sqrt[3]{a^2-b}=x^2-y$. If $c=\sqrt[3]{a^2-b}$, $x^2-y=c$ . . . . (1).

Cubing both sides of $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, we have $a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$. ∴ $a=x^3+3xy=x^3+3x(x^2-c)$, ∵ $y=x^2-c$ from (1). ∴ $4x^3-3cx=a$. From this x must be found *by trial* and then y is known from $y=x^2-c$.

Note.—Unless a^2-b be a *perfect cube*, the method is inapplicable.

Ex. Find the cube root of $7+5\sqrt{2}$.

Let $\sqrt[3]{(7+5\sqrt{2})}=x+\sqrt{y}$...(1) Then $\sqrt[3]{(7-5\sqrt{2})}=x-\sqrt{y}$.

∴ $\sqrt[3]{(7^2-25\times 2)}=x^2-y$. ∴ $x^2-y=\sqrt[3]{-1}=-1$. ∴ $y=x^2+1$ . . . (2). Cubing (1) and equating the rational parts, $7=x^3+3xy$. ∴ $x^3+3x(x^2+1)=7$. ∴ $4x^3+3x=7$. By trial, the value of x must be found. $x=1$ satisfies the relation $4x^3+3x=7$. Hence $y=2$. ∵ $y=x^2+1$. ∴ the cube root required is $1+\sqrt{2}$.

EXERCISE 53.

Find the rationalising factors of :—

1. $1+\sqrt[3]{2}$. 2. $a^{\frac{1}{2}}-b^{\frac{2}{3}}$. 3. $\sqrt{2}-\sqrt[3]{2}$. 4. $\sqrt{3}+\sqrt[4]{5}$.

5. $\sqrt{2}-\sqrt[4]{3}$. 6. $a-\sqrt[4]{b}$. 7. $\sqrt{5}+\sqrt[4]{7}$. 8. $2\sqrt{3}-3\sqrt[3]{2}$.

9. Find the value of $x^3-3qx-2r$, when $x=\sqrt[3]{(r+\sqrt{r^2-q^3})}+\sqrt[3]{(r-\sqrt{r^2-q^3})}$.

10. Find the value of $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1-\sqrt{1-x}}$ when $x=\dfrac{\sqrt{3}}{2}$.

11. Find the value of $\dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$ when $x=\frac{1}{2}\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right)$.

Find the square roots of :—

12. $7+4\sqrt{3}$. 13. $16-6\sqrt{7}$. 14. $43+12\sqrt{7}$.

15. $2a+2\sqrt{a^2-b^2}$. 16. $4-\sqrt{15}$. 17. $4+2\sqrt{3}$.

18. $-9+6\sqrt{3}$. 19. $75-12\sqrt{21}$.

20. $ab+c^2+\sqrt{(a^2-c^2)(b^2-c^2)}$.

21. Find the value of $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1+\sqrt{1-x}}$ when $x=\dfrac{\sqrt{3}}{2}$.

22. Find the square root of $6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$.

23. Find the square root of $9+2\sqrt{6}-4\sqrt{3}-4\sqrt{2}$.

24. Find the square root of $5+\sqrt{10}-\sqrt{6}-\sqrt{15}$.

Find the cube roots of :—

25. $10+6\sqrt{3}$. 26. $16+8\sqrt{5}$. 27. $9\sqrt{3}-11\sqrt{2}$.

28. $7-5\sqrt{2}$. 29. $26+15\sqrt{3}$. 30. $45+29\sqrt{2}$.

31. $20+14\sqrt{2}$. 32. $1+x\sqrt{x}+3x+3\sqrt{x}$.

33. Simplify $\dfrac{1}{\sqrt{(11-2\sqrt{30})}}-\dfrac{3}{\sqrt{(7-2\sqrt{10})}}-\dfrac{4}{\sqrt{(8+4\sqrt{3})}}$.

34. Prove that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}=\sqrt{2}$.

35. Shew that $\sqrt[3]{(\sqrt{5}+2)}-\sqrt[3]{(\sqrt{5}-2)}=1$.

CHAPTER XX.

RATIO AND PROPORTION.

129. Definitions.

Ratio. The term *ratio* is usually defined to be the relation between two quantities of the *same* kind in regard to their magnitude, comparison being made by considering what multiple, part, or parts, one is of the other.

The ratio of a to b is usually written $a : b$, and is measured by the fraction $\frac{a}{b}$.

a and b are called the **terms** of the ratio. The first term a is called the **antecedent**, the second term b is called the **consequent.**

If $a=b$, then $a : b$ is called a **ratio of equality.**

If $a>b$, then $a : b$ is called a **ratio of greater inequality.**

If $a<b$, then $a : b$ is called a **ratio of less inequality.**

The ratio of $a^2 : b^2$ is called the **duplicate ratio** of $a : b$.

The ratio ot $a^3 : b^3$ is called the **triplicate ratio** of $a : b$.

The ratio of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **subduplicate ratio** of $a : b$.

The ratio of $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is called the **subtriplicate ratio** of $a : b$.

If the ratio of two quantities can be expressed exactly by the ratio of two integers, the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.** Thus the ratio of $\sqrt{2} : \sqrt{3}$ cannot be exactly expressed by two integers; therefore they are *incommensurable* quantities.

Proportion. When two ratios are equal, the four quantities composing them are said to be **proportionals.** Thus, if $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d are proportionals. This is expressed by saying that a is to b as c is to d, and the proportion is written $a : b :: c : d$; or $a : b=c : d$.

The terms a and d are called the *extremes*, b and c the *means.*

Continued Proportion. If $a : b = b : c$, then a, b, c are in continued proportion. Here b is said to be the *mean proportional* between a and c; and c is said to be the third proportional to a and b. If $a : b = b : c = c : d$, then a, b, c, d are said to be in continued proportion.

130. Comparison of Ratios. Two or more ratios may be compared by reducing their equivalent fractions to a common denominator. Thus the ratios $a : b$ and $c : d$ are compared by reducing them to the equivalent forms $\frac{ad}{bd}$ and $\frac{cb}{bd}$. The ratio $a : b$ is $=$ or $>$ or $<$ the ratio $c : d$ according as ad is $=$ or $>$ or $<$ cb.

Approximation of Ratios. Let $a + x : a$ be a ratio, where x is *very small* compared with a; then the ratio of $(a+x)^2 : a^2$ is approximately equal to the ratio of $a + 2x : a$.

We have $\frac{(a+x)^2}{a^2} = \frac{a^2 + 2ax + x^2}{a^2} = 1 + \frac{2x}{a} + \frac{x^2}{a^2}$.

Since x is very small compared with a, the value of $\frac{x^2}{a^2}$ is very small compared with $\frac{2x}{a}$ and 1; and therefore it may be neglected; $\therefore$ approximately $\frac{(a+x)^2}{a^2} = 1 + \frac{2x}{a} = \frac{a+2x}{a}$. $\therefore (a+x)^2$ $a^2 = a + 2x : a$ (nearly).

Similarly $(a+x)^3 : a^3 = a + 3x : a$ (nearly) and $(a+x)^{\frac{1}{2}} : a^{\frac{1}{2}} = a + \frac{1}{2}x : a$ (nearly).

131. (1) If $a > b$, then $\frac{a}{b} > \frac{a+x}{b+x}$. [$a, b, x$ are pos.]

Since $a > b$, $\therefore ax > bx$. $\therefore ax + ab > bx + ab$.

$\therefore a(x+b) > b(x+a)$ $\therefore \frac{a}{b} > \frac{x+a}{x+b}$, i.e., $\frac{a+x}{b+x}$.

(2) If $a < b$, then $\frac{a}{b} < \frac{a+x}{b+x}$. [$a, b, x$ are pos.]

Since $a < b$, $\therefore ax < bx$. $\therefore ax + ab < bx + ab$, $\therefore a(x+b) < (x+a)$. $\therefore \frac{a}{b} < \frac{x+a}{x+b}$, i.e., $\frac{a+x}{b+x}$.

From (1) and (2) we infer that *a ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

Similarly it can be proved that if $a>b$, $\frac{a}{b}<\frac{a-x}{b-x}$; and that if $a<b$, $\frac{a}{b}>\frac{a-x}{b-x}$. That is, *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting the same quantity from both its terms.*

132. (I) If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$. That is, *when four quantities are proportionals, the product of the extremes is equal to the product of the means.*

(II) If $\frac{a}{b}=\frac{c}{d}$, then (1) $\frac{a+b}{b}=\frac{c+d}{d}$. [Add 1 to both sides of $\frac{a}{b}=\frac{c}{d}$ and simplify]; (2) $\frac{a-b}{b}=\frac{c-d}{d}$. [Subtract 1 from both sides of $\frac{a}{b}=\frac{c}{d}$ and simplify] and (3) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. [Divide (1) by (2)].

133. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each of the ratios $=\left(\frac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$ where p, q, r, n are any quantities whatever.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=kb$; $c=kd$ and $e=kf$.

$\therefore p\,.\,a^n=p(kb)^n$; $qc^n=q(kd)^n$ and $re^n=r(kf)^n$.

$\therefore p\,.\,(kb)^n+q(kd)^n+r(kf)^n=pa^n+qc^n+re^n$.

$\therefore k^n=\frac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}$. $\qquad \therefore k=\left(\frac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$.

Note.—By giving to p, q, r, n different values, many particular cases may be deduced; or they may be proved by the same method. If $p=q=r=n=1$, then each of the ratios $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}=\frac{a+c+e}{b+d+f}$. That is, *when a number of fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

The same may be proved independently thus :—

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $kb=a$; $kd=c$ and $kf=e$.

$\therefore kb+kd+kf=a+c+e$ $\quad\therefore k(b+d+f)=a+c+e$.

$\therefore k=\frac{a+c+e}{b+d+f}$.

Ex. 1. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)\times(a+b-c-d)$, shew that $a:b=c:d$.

From the given relation we have $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$.

$\therefore \frac{a+b+c+d+a+b-c-d}{a+b+c+d-a-b+c+d}=\frac{a-b+c-d+a-b-c+d}{a-b+c-d-a+b+c-d}$.

[II of Art. 132].

$\therefore \frac{a+b}{c+d}=\frac{a-b}{c-d}$. $\quad\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d}$.

$\therefore \frac{a+b+a-b}{a+b-a+b}=\frac{c+d+c-d}{c+d-c+d}$ ... [II of Art 132]

$\therefore \frac{a}{b}=\frac{c}{d}$, i.e., $a : b=c : d$.

Ex. 2. If $x=\frac{2ab}{b^2+1}$, find the value of $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

Since $x=\frac{2ab}{b^2+1}$. $\quad\therefore \frac{x}{a}=\frac{2b}{b^2+1}$. $\quad\therefore \frac{a}{x}=\frac{b^2+1}{2b}$.

$\therefore \frac{a+x}{a-x}=\frac{b^2+1+2b}{b^2+1-2b}$. [II of Art 132] $=\frac{(b+1)^2}{(b-1)^2}$.

$\therefore \frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{b+1}{b-1}$. $\quad\therefore \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\frac{b+1+b-1}{b+1-b+1}=b$.

Ex. 3. If $x=\frac{\sqrt[3]{p+1}+\sqrt[3]{p-1}}{\sqrt[3]{p+1}-\sqrt[3]{p-1}}$, shew that $x^3-3px^2+3x-p=0$.

Since $x=\frac{\sqrt[3]{p+1}+\sqrt[3]{p-1}}{\sqrt[3]{p+1}-\sqrt[3]{p-1}}$, $\quad\therefore \frac{x+1}{x-1}=\frac{\sqrt[3]{p+1}}{\sqrt[3]{p-1}}$.

[II of Art 132]

$$\therefore \frac{(x+1)^3}{(x-1)^3}=\frac{p+1}{p-1}, \quad \therefore \frac{(x+1)^3+(x-1)^3}{(x+1)^3-(x-1)^3}=\frac{p+1+p-1}{p+1-p+1}.$$

[II of Art. 132].

$$\therefore \frac{x^3+3x}{3x^2+1}=p. \therefore x^3+3x=3px^2+p. \therefore x^3-3px^2+3x-p=0.$$

Ex. 4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that $\frac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\frac{ace}{bdf}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk$, $c=dk$, $e=fk$.

$\therefore a^3b=b(b^3k^3)=b^4k^3$; $2c^2e=2\,.\,(dk)^2\,.\,fk=2d^2fk^3$.

$-3ae^2f=-3(bk)(fk)^2\,.\,f=-3bf^3k^3$.

$\therefore$ by adding $k^3(b^4+2d^2f-3bf^3)=a^3b+2c^2e-3ae^2f$.

$$\therefore \quad k^3=\frac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3} \text{ and } k^3=\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ace}{bdf}.$$

$$\therefore \quad \frac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\frac{ace}{bdf}.$$

Ex. 5. If $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$, shew that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Multiplying the terms of the given ratios by c, b, a respectively, we have $\frac{acy-bcx}{c^2}=\frac{bcx-abz}{b^2}=\frac{abz-acy}{a^2}$.

$$\therefore \text{ each } = \frac{acy-bcx+bcx-abz+abz-acy}{c^2+b^2+a^2} \text{ [Art. 133]}=0.$$

$$\therefore \quad \frac{acy-bcx}{c^2}=0. \quad \therefore \frac{ay-bx}{c}=0. \quad \therefore ay=bx. \quad \therefore \frac{x}{a}=\frac{y}{b}.$$

Again $\frac{bcx-abz}{b^2}=0. \quad \therefore \frac{cx-az}{b}=0, \quad \therefore cx=az. \quad \therefore \frac{x}{a}=\frac{z}{c}.$

$$\therefore \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Ex. 6. If $\frac{a}{b}=\frac{c}{d}$, shew that $\frac{ma+nc}{mb+nd}=\sqrt{\frac{a^2+c^2}{b^2+d^2}}$.

Since $\frac{a}{b}=\frac{c}{d}$, $\therefore \frac{ma}{mb}=\frac{nc}{nd}$.

$\therefore$ each $=\frac{ma+nc}{mb+nd}$. . . . (1) [Art. 133].

Again since $\frac{a}{b}=\frac{c}{d}$, $\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}$.

$\therefore$ each of these $=\frac{a^2+c^2}{b^2+d^2}$. . . . (2) [Art. 133].

We have $\frac{ma+nc}{mb+nd}=\frac{a}{b}$ and $\frac{a^2+c^2}{b^2+d^2}=\frac{a^2}{b^2}$.

$$\therefore \quad \frac{ma+nc}{mb+nd}=\sqrt{\frac{a^2+c^2}{b^2+d^2}}.$$

EXERCISE 54.

1. If $x=\frac{4ab}{a+b}$, find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$.

2. If $x=\frac{\sqrt[3]{a+1}+\sqrt[3]{a-1}}{\sqrt[3]{a+1}-\sqrt[3]{a-1}}$, find the value of $\frac{x^3+3x}{3x^2+1}$.

3. If $x=\frac{\sqrt{m+1}+\sqrt{m-1}}{\sqrt{m+1}-\sqrt{m-1}}$, shew that $x^2-2mx+1=0$.

4. If $x=\frac{\sqrt{2p+3q}+\sqrt{2p-3q}}{\sqrt{2p+3q}-\sqrt{2p-3q}}$, shew that $3qx^2-4px+3q=0$.

5. If $\frac{\sqrt{x^2+y^2}-\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}+\sqrt{x^2-y^2}}=m$, shew that $\frac{1+m^2}{2m}=\frac{x^2}{y^2}$.

6. If $x=\frac{\sqrt[4]{a+1}+\sqrt[4]{a-1}}{\sqrt[4]{a+1}-\sqrt[4]{a-1}}$, shew that $x^4-4ax^3+6x^2-4ax+1=0$.

7. If $(a+3b+2x+6y)(a-3b-2x+6y)=(a-3b+2x-6y)\times(a+3b-2x-6y)$, shew that $\frac{a}{b}=\frac{x}{y}$.

8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that (i) $\frac{a}{b}=\left(\frac{a^2-ac+e^2}{b^2-bd+f^2}\right)^{\frac{1}{2}}$;

(ii) $\frac{a}{b}=\left(\frac{a^3-3ace+e^3}{b^3-3bdf+f^3}\right)^{\frac{1}{3}}$; (iii) $\frac{a}{b}=\frac{3a-4c+5e}{3b-4d+5f}$.

9. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that $\sqrt{(a+c+e)(b+d+f)}=\sqrt{ab}+\sqrt{cd}+\sqrt{ef}$.

10. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, shew that $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

11. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, shew that (i) $\frac{a}{b+d}=\frac{c^3}{c^2d+d^3}$;

(ii) $\frac{2a+3d}{3a-4d}=\frac{2a^3+3b^3}{3a^3-4b^3}$.

12. If $\frac{a}{b}=\frac{c}{d}$, shew that $\frac{a-c}{b-d}=\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$.

13. If $\frac{a}{b}=\frac{b}{c}$, shew that $b^4=\frac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}}$.

14. If $\frac{a}{b}=\frac{c}{d}$, shew that $\frac{a^2+b^2}{c^2+d^2}=\frac{a^3(c+d)}{c^3(a+b)}$.

15. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, shew that $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$.

16. If $ad=bc$, prove the following identities:—

(i) $\left(\frac{a+c}{b+d}\right)=\frac{a(a-c)^2}{b(b-d)^2}$; (ii) $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$;

(iii) $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$; (iv) $\frac{a^2+ab}{c^2+cd}=\frac{b^2-2ab}{d^2-2cd}$.

17. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, shew that $(b-c)x+(c-a)y+(a-b)z=0$.

18. If $\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}$, prove that $\frac{a(b-c)}{y^2-z^2}=\frac{b(c-a)}{z^2-x^2}=\frac{c(a-b)}{x^2-y^2}$.

19. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, prove that $\dfrac{x^2+a^2}{x+a}+\dfrac{y^2+b^2}{y+b}+\dfrac{z^2+c^2}{z+c}$

$$=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}.$$

20. If $\dfrac{a}{q+r-p}=\dfrac{b}{r+p-q}=\dfrac{c}{p+q-r}$, shew that

$$(q-r)a+(r-p)b+(p-q)c=0.$$

21. If $\dfrac{x}{ax+by+cz}=\dfrac{y}{bx+cy+az}=\dfrac{z}{cx+ay+bx}$, then each

$$=\frac{1}{a+b+c}.$$

22. If $\dfrac{a+b}{a-b}=\dfrac{b+c}{2(b-c)}=\dfrac{c+a}{3(c-a)}$, then $8a+9b+5c=0$.

23. If $\dfrac{mx-a-b}{nx-c-d}=\dfrac{mx-a-c}{nx-b-d}$, then each $=-1$.

24. If $x(a+x-b)=y(b+y-c)=z(c+z-a)$, then each

$$=\frac{x+y+z}{x^{-1}+y^{-1}+z^{-1}}.$$

25. If $\dfrac{ad-bc}{a-b-c+d}=\dfrac{ac-bd}{a-b-d+c}$, then each $=\frac{1}{4}(a+b+c+d)$.

26. If b and c are unequal and $\dfrac{b^2-ac}{a+c-2b}=\dfrac{c^2-bd}{b+d-2c}$, then each

$$=\frac{bc-ad}{a+d-b-c}.$$

CHAPTER XXI.

EXAMINATION PAPERS.

THIRD SERIES ON CHAPTERS XVI—XX.

I.

1. Write down the last four terms in the expansion of $(x-y)^{50}$, $(3a-2b)^{45}$ and $(a-b)^{100}$.

2. If a number contains n digits, how many digits will its *square* contain? How many will its *cube* contain?

3. Shew that the following expressions are *perfect* squares.

(a) $(a+b+c)^2-4(ac+bc)$; (b) $(ab+ac+bc)^2-4abc(a+c)$;

(c) $2\{(x-y)^4+(y-z)^4+(z-x)^4\}$.

4. Shew that $\left\{\dfrac{-1+\sqrt{-3}}{2}\right\}^3+\left\{\dfrac{-1-\sqrt{-3}}{2}\right\}^3=2.$

5. Extract the square roots of (i) $n^2+1+\sqrt{n^4+n^2+1}$; (ii) $n^2+2-\sqrt{n^4+4}$; (iii) $52-14\sqrt{3}$; (iv) $4^n+2^{n+1}+1$; (v) $15+2\sqrt{15}+2\sqrt{35}+2\sqrt{21}$.

6. Extract the cube roots of (i) $20+14\sqrt{2}$; (ii) $a^3+3a^2\sqrt{b}+3ab+b\sqrt{b}$; (iii) $(a+b)^3+(a-b)^3+6a(a^2-b^2)$.

7. Rationalise (i) $\dfrac{x+\sqrt{1+x^2}}{x-\sqrt{1-x^2}}$; (ii) $\dfrac{\sqrt[4]{a+b}+\sqrt[4]{a-b}}{\sqrt[4]{a+b}-\sqrt[4]{a-b}}$.

8. Resolve into factors (i) $a^{-4}+a^{-2}b^{-2}+b^{-4}$; (ii) $x+y-z+3\sqrt[3]{xyz}$; (iii) $(x+y)^{\frac{3}{2}}-z(x+y)^{\frac{1}{2}}+2(x^2y+xy^2)^{\frac{1}{2}}$.

9. If $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{b}{a}+\dfrac{d}{c}$, shew that $\dfrac{a^n}{b^n}+\dfrac{c^n}{d^n}=\dfrac{b^n}{a^n}+\dfrac{d^n}{c^n}$.

10. Simplify $\dfrac{a+(a^2-b^2)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}}+\dfrac{a-(a^2-b^2)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}-(a-b)^{\frac{1}{2}}}$.

II.

1. Find the continued product of—

(a) $\sqrt{a^2+b^2}+a+b$, $\sqrt{a^2+b^2}-a+b$, $\sqrt{a^2+b^2}+a-b$ and $a+b-\sqrt{a^2+b^2}$; (b) $\sqrt{(a^2-1)(x^2-1)}+ax-1$, $\sqrt{(a^2-1)(x^2-1)}-ax+1$, $\sqrt{(a^2-1)(x^2-1)}+ax+1$ and $\sqrt{(a^2-1)(x^2-1)}-ax-1$.

2. Extract the square root of $x^2-4a\sqrt{b}+a^2-2ax+4x\sqrt{b}+4b$ and the cube root of $a^{3x}+3a^{2x}+6a^x+7+6a^{-x}+3a^{-2x}+a^{-3x}$.

3. Shew that the following expressions are *perfect* squares.

(a) $\left\{\dfrac{(a+b)(a+c)-bc}{a}\right\}\left\{\dfrac{(a+b)(b+c)-ac}{b}\right\}$;

(b) $4a^2b^2-2ab(a^2+1)(b^2+1)+(a^2+b^2)(a^2b^2+1)$;

(c) $\dfrac{a}{x^2}+\dfrac{b}{y^2}+\dfrac{2ab}{xy}$ if $ab=1$;

(d) $\{(a^2+b^2)-(c^2+d^2)\}^2+4\{(ac+bd)^2+(ad-bc)^2\}$.

4. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$, then shew that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

5. Find the *ninth* root of $\left(x^3+\dfrac{1}{x^3}\right)^2\left\{9\left(x+\dfrac{1}{x}\right)+x^2+\dfrac{1}{x^2}\right\}+27\left(x+\dfrac{1}{x}\right)^2\left\{\left(x+\dfrac{1}{x}\right)+\left(x^3+\dfrac{1}{x^3}\right)\right\}$.

6. Find the L.C.M. of $x-a$, $x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}$, $x^{\frac{4}{3}}+x^{\frac{2}{3}}a^{\frac{2}{3}}+a^{\frac{4}{3}}$ and $x^{\frac{2}{3}}-a^{\frac{2}{3}}$, and the H.C.F. of $x^{\frac{3}{2}}y^{\frac{1}{2}}+xy+x+x^{\frac{1}{2}}yz^{\frac{1}{2}}+y^{\frac{1}{2}}z^{\frac{1}{2}}-1$ and $x^{\frac{3}{2}}y^{\frac{1}{2}}+xy+x-x^{\frac{1}{2}}yz^{\frac{1}{2}}-y^{\frac{1}{2}}z^{\frac{1}{2}}-1$.

7. If $a^x=m$, $a^y=n$ and $a^{\frac{2}{z}}=m^yn^x$, prove that $xyz=1$.

8. If $\dfrac{a}{b}=\dfrac{p}{q}$, then (1) $a^2+b^2:\dfrac{a^3}{a+b}=p^2+q^2:\dfrac{p^3}{p+q}$.

(2) $ma+nb:ca+db=mp+nq:cp+dq$.

9. If $\dfrac{x+y}{x-y}=\dfrac{m}{n}$, shew that $\dfrac{x^2+y^2}{x^2-y^2}=\dfrac{m^2+n^2}{2mn}$.

If $x=\sqrt{ac}$, shew that $x\,\frac{x+3a}{c+3x}=\sqrt{(ac)}\,\frac{a+3x}{x+3c}$.

10. If $\frac{x^2-yz}{x(1-yz)}=\frac{y^2-xz}{y(1-xz)}$, then each will be equal to $x+y+z$.

III.

1. If $c=a\sqrt{1-b^2}+b\sqrt{1-a^2}$, find the value of—

$$(a+b+c)(a+b-c)(c+a-b)(b+c-a).$$

2. Write down the co-efficient of x^5 in the expansion of $(1+2x-x^2)^7$ and of x^4 in the expansion of $(1-ax+bx^2)^4$.

3. Find the square roots of (i) $\frac{a^4}{a^2-x}$ to three terms,

(ii) $a^{4m}-4a^{3m+n}+6a^{2(m+n)}-4a^{m+3n}+a^{4n}$,

(iii) $\frac{9x^3}{4}-5x^{\frac{5}{2}}y^{\frac{1}{2}}+\frac{179x^2y}{45}-\frac{4x^{\frac{3}{2}}y^{\frac{3}{2}}}{3}+\frac{4xy^2}{25}$.

4. Simplify (a) $\frac{2\sqrt{2}+\sqrt{3}-1}{\sqrt{3}+1}-\frac{\sqrt{2}-1}{\sqrt{2}+\sqrt{3}}-\frac{(\sqrt{2}-1)(2\sqrt{2}+\sqrt{3}-1)}{(\sqrt{2}+\sqrt{3})(\sqrt{3}+1)}$.

(b) $\frac{\sqrt{3}}{2-\sqrt{3}}\sqrt{\left\{\frac{4+\sqrt{3}-2\sqrt{2}\sqrt{(2+\sqrt{3})}}{4+\sqrt{3}+2\sqrt{2}\sqrt{(2+\sqrt{3})}}\right\}}$.

5. If $\frac{3x+2y}{4a+3b}=\frac{3y+2z}{4b+3c}=\frac{3z+2x}{4c+3a}$, shew that $5(x+y+z)\times(20a+18b+25c)=7(a+b+c)(14x+13y+18z)$.

6. If $\frac{a}{b}=\frac{c}{d}$, shew that—

(i) $(a-nc)^2 \,.\, (b-nd)^2=a^2+c^2 : b^2+d^2=a^2-c^2 : b^2-d^2$;

(ii) $\frac{2a+3b+4c+6d}{2a+3b-4c-6d}=\frac{2a-3b+4c-6d}{2a-3b-4c+6d}$.

7. Find the value of $mn-\sqrt{(1-m^2)(1-n^2)}$ when $2m=x+\frac{1}{x}$, $2n=y+\frac{1}{y}$.

8. Find the cube root of $a^6-3a^5+9a^4-13a^3+18a^2-12a+8$.

9. Find the value of $\{x^2-2xy+4y^2\}^{\frac{1}{2}}$ in terms of a and b where $x=9a^2-12ab$ and $y=2b^2-6ab$.

10. Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$ when x

$$=\frac{12ab}{a+b+\sqrt{\{(a+b)^2+12ab\}}}.$$

IV.

1. Shew that $a^{-2}=\dfrac{1}{a^2}$ — and $a^{\frac{1}{p}}=\sqrt[p]{a}$.

Simplify $\dfrac{(1-x)^{-1}+(1+x)^{-1}}{(1-x)^{-1}-(1+x)^{-1}}$.

2. Express $\sqrt{\sqrt[3]{(a+b)}}\times\sqrt{\sqrt[3]{(a-b)}}$ as a single surd.

3. Find the value of $\dfrac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{(1+a)}\sqrt{(1-a)}}$ when $a=\dfrac{\sqrt{3}}{2}$.

4. Find a value of x which will make the expression $x^5-8x^3+11x^2+7x-1789$, exactly divisible by x^2+7x-1.

5. If $(x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})$ $=(x-\sqrt{x^2-bc})(y-\sqrt{y^2-ca})(z-\sqrt{z^2-ab})$, shew that each is equal to $\pm abc$.

6. If $ab=cd=ef$, then $\dfrac{ac+ce+ea}{bdf(b+d+f)}=\dfrac{a^2+c^2+e^2}{d^2f^2+f^2b^2+b^2d^2}$.

7. If $a^4-6a^3+23a^2$ be the first three terms of the square of a trinomial, find the last two terms, and the root.

8. What value of a makes $a^6-6a^5+15a^4-20a^3+15a^2-7a+8$ a complete cube?

9. If the difference of two numbers $=1$, then the difference of their squares $=$ their sum.

10. If $A(x-3)(x-5)+B(x-5)(x-7)+C(x-7)(x-3)=8x-120$, for *all* values of x, determine the values of A, B, C.

V.

1. Shew that $a^m\times a^n=a^{m+n}$; and thence deduce $a^0=1$; and $a^{-n}=\dfrac{1}{a^n}$.

2. If $a+\sqrt{b}=x+\sqrt{y}$, prove that $a=x$ and $b=y$.
Extract the cube root of $100+51\sqrt{3}$.

3. Extract the square root of $x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+a^{\frac{8}{5}}$ and shew that $a^4+b^4+c^4+d^4-2(a^2+c^2)\times(b^2+d^2)+2a^2c^2+2b^2d^2$ is an exact square.

4. If $x=\frac{1}{a}\sqrt{\left(\frac{2a}{b}-1\right)}$, find the value of $\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}$.

5. If $x=\frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$, find the value of $\frac{x+\sqrt{8}}{x-\sqrt{8}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}$.

6. If $a:b=b:c$, shew that $a^4+a^2c^2+c^4=b^2\left(\frac{b^2}{c^2}+\frac{b^2}{a^2}-1\right)\times(a^2+b^2+c^2)$.

7. If $(1+ab+cd)^2=(1+a^2+c^2)(1+b^2+d^2)$, shew that $a=b$ and $c=d$.

8. Simplify (i) $\frac{\sqrt{x^2-y^2}+x}{\sqrt{x^2+y^2}+y}\div\frac{\sqrt{x^2+y^2}-y}{x-\sqrt{x^2-y^2}}$;

(ii) $\frac{1}{2\sqrt{7}-3\sqrt{2}}-\frac{1}{2\sqrt{7}+3\sqrt{2}}$.

9. If $a^6+9a^5+12a^4-63a^3$ be the first four terms of the cube of a trinomial, find the last three terms and the root.

10. Find the value of $\sqrt{4q^3-27r^2}$, when $q=3(1+a+a^2)$ and $r=1+3a+3a^2+2a^3$.

VI.

1. Find the co-efficient of x in the expansion of—
$$(x-a)^2(x-b)^2(x-c)^2.$$

2. Prove that the sum of the cubes of any three *consecutive* integers is divisible by three times the mean integer.

3. Find the conditions that px^3+qx^2+rx+s should be a *perfect* cube.

4. Find the rationalising factor of $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

5. If $x=\frac{2ac}{b(1+c^2)}$, find the value of $\frac{\sqrt{a+bx}+\sqrt{a-bx}}{\sqrt{a+bx}-\sqrt{a-bx}}$.

6. If $x=\sqrt[3]{\left\{-\frac{1}{2}r+\sqrt{\left(\frac{1}{4}r^2-\frac{1}{27}q^3\right)}\right\}}$, find the value of $x^6+rx^3+\frac{1}{27}q^3$.

7. Find the square root of (i) $a+1-2\sqrt{a}(1+\sqrt{x})+3\sqrt{x}$;

(ii) $1+m^3+2(1-m^2)\sqrt{m}+3m-m^2$.

(iii) $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}-2x^{-\frac{1}{6}}+x^{-\frac{1}{3}}-1$.

8. If $x^{-1}=(a-c)(b-c)$, $y^{-1}=(a-b)(b-c)$, $z^{-1}=(a-b)(a-c)$, find the value of $x-y+z$ and $abx-acy+bcz$.

9. If $\frac{a}{b}=\frac{b}{c}$, shew that $\frac{a+b}{b+c}=\frac{a^2(b-c)}{b^2(a-b)}$.

10. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, shew that $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$.

VII.

1. Find the square roots of—

(a) $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$. (b) $15-8\sqrt{2}+4\sqrt{3}-4\sqrt{6}$.

(c) $\frac{17}{3}-4\sqrt{2}$. (d) $97+28\sqrt{12}$.

2. Find the value of $\frac{2a\sqrt{1+y^2}}{y+\sqrt{1+y^2}}$, when $y=\frac{1}{2}\left\{\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right\}$.

3. Find the relation between a, b, c and d, when $x^4+ax^3+bx^2+cx+d$, is a *perfect* square.

4. Shew that $x^n-na^{n-1}x+(n-1)a^n$ is divisible by $(x-a)^2$ if n be a whole number.

5. If $\frac{a}{b}=\frac{b}{c}$, shew that $a-2b+c=\frac{(a-b)^2}{a}=\frac{(b-c)^2}{c}$ and that $a^3+b^3+c^3=a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$.

6. Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the *difference* of two squares.

7. What value of x will make $x^4+6x^3+11x^2+3x+31$ a *perfect* square?

8. Shew that $(a^2-bc)^3+(b^2-ac)^3+(c^2-ab)^3-3(a^2-bc)\times(b^2-ac)(c^2-ab)$ is an *exact* square.

9. If $x^2=4+2\sqrt{3}$, $y^2=7+4\sqrt{3}$, $z^2=21+12\sqrt{3}$, find the value of $x^3+y^3-z^3+3xyz$.

10. If $a+b+c=0$, shew that $a^7+b^7+c^7=7abc(c^2-ab)^2$.

VIII.

1. Find the L. C. M. of ax^2-1, ax^2+1, $(a^{\frac{1}{2}}x-1)^2$, $(a^{\frac{1}{2}}x+1)^2$, $a^{\frac{3}{2}}x^3-1$ and $a^{\frac{3}{2}}x^3+1$.

2. Simplify $\{(a-b)^2+4ab\}^{\frac{1}{2}}\times\{(a+b)^2-4ab\}^{\frac{3}{2}}$

$$\times\left\{\frac{a^4-b^4}{a-b}+2ab(a+b)\right\}^{\frac{2}{3}}.$$

3. Divide $a+b^2+c^3-3\sqrt[3]{ab^2c^3}$ by $a^{\frac{1}{3}}+b^{\frac{2}{3}}+c$.

4. Find the *fourth* root of—

$$\frac{1}{16}x^{\frac{20}{3}}-\frac{5}{2}x^5y^{\frac{4}{5}}+\frac{75}{2}x^{\frac{10}{3}}y^{\frac{8}{5}}-250x^{\frac{5}{3}}y^{\frac{12}{5}}+625y^{\frac{16}{5}}.$$

5. Reduce to their lowest terms—

(*a*) $\dfrac{x^3+a^2x^2-ax-a^3}{x^4-ax+a^{\frac{1}{2}}x-a^{\frac{3}{2}}}$. (*b*) $\dfrac{3ax^3-2a^{\frac{2}{3}}x^2-a^{\frac{1}{3}}x}{6a^{\frac{2}{3}}x^2-a^{\frac{1}{3}}x-1}$.

6. If $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$, find the value of x^2+xy+y^2 and x^3+y^3.

7. If $\dfrac{x}{a+2b+c}=\dfrac{y}{2a+b-c}=\dfrac{z}{4a-4b+c}$, shew that—

$$\frac{a}{x+2y+z}=\frac{b}{2x+y-z}=\frac{c}{4x-4y+z}.$$

8. If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a^2}{b^2}+\dfrac{c^2}{d^2}=\dfrac{2ac}{bd}$.

9. If $(a^2-bc)(b^2-ac)(c^2-ab)=0$, shew that—

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^2b^2c^2}.$$

10. If a number contains n digits, how many digits will its *square root* contain? How many will its *cube root* contain?

IX.

1. If A contain p digits and B contain q digits, how many digits will $A \times B$ contain?

2. If $\dfrac{a^2}{bc} = \dfrac{b^2}{ac} = \dfrac{c^2}{ab} = 1$, then $a(a^2+bc)+b(b^2+ac)+c(c^2+ab) - 6abc = 0$.

If $x = \dfrac{1}{2}(p^n + q^n)$, find the value of $\dfrac{p^n}{2np^n - 2nx} + \dfrac{q^n}{2nq^n - 2nx}$.

3. If $x^4 + px^3 + qx^2 + p + 1$ be a *complete* square, then $p^2 = 4q - 8$.

4. Find the product of $(1+a+a^2+ \ldots\ldots\ldots + a^{2n})$ and $(1-a+a^2-a^3+a^4- \ldots + a^{2n})$ without actual multiplication.

5. Resolve into factors (i) $1 - a^{\frac{1}{2}} - a + a^{\frac{3}{2}}$;

(ii) $a^{-\frac{3}{2}} + b^{-1} + c^{-3} - 3a^{-\frac{1}{2}} b^{-\frac{1}{3}} c^{-1}$.

6. If $x + a$ be a factor of $a^2x^3 - b^3x^2 + ac^3x + 3a^3bc$, and if a is not equal to zero, prove that $a^3 + b^3 + c^3 = 3abc$.

7. Divide (i) $2yz + 2zx + 2xy - x^2 - y^2 - z^2$ by $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y - z$;

(ii) $x^{\frac{8}{3}} + x^{\frac{4}{3}} + 1$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$.

8. Reduce to its simplest form—

$$\left(x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}}\right)\left(x^{\frac{1}{4}} + \frac{1}{x^{\frac{1}{4}}}\right)\left(x^{\frac{1}{8}} + \frac{1}{x^{\frac{1}{8}}}\right)\left(x^{\frac{1}{8}} - \frac{1}{x^{\frac{1}{8}}}\right).$$

9. Shew that $(x+y)^5 - x^5 - y^5$ is divisible by $(x^2 + xy + y^2)$.

10. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that $(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) = (ax + by + cz)^2$.

X.

1. What is the difference between $a^{x^{y^z}}$ and $\{(a^x)^y\}^z$ if $x = 2$, $y = z = 3$? Shew that $\dfrac{(xy^2)^{\frac{1}{3}} - (x^2y)^{\frac{1}{3}} + x}{x + y} = \dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}$.

2. Find the co-efficient of x^7 in $(x-2a)^9$ and of x^8 in $(3x-a)^{11}$.

3. What value of y will make $4y^4+36y^3+85y^2+16y+5$ a *complete* square? What must be added to $(ax+b)(ax-3b)\times(ax-b)(ax+3b)$ to make it a *perfect* square?

4. Find the *fourth* root of $\left(a^2-6+\frac{1}{a^2}\right)^2+\left(4a-\frac{4}{a}\right)^2$.

5. What value of x will make $x^3+3cx^2+2c^2x+5c^3$
$$=(x+c)^3\ ?$$

6. Express $2x^2y^2+\frac{2}{x^2y^2}$ as the *sum* of two squares.

7. If $a^2+b^2=1$, $2ab=\frac{c-d}{c+d}$ and $a+b=2an$; shew that—
$$\left(1-\frac{1}{n}\right)^2=\frac{d}{c}.$$

8. If $x=\frac{b}{a}\sqrt{a^2+b^2}$, find $\frac{\sqrt{x^2+2bx+a^2}+\sqrt{x^2-2bx+a^2}}{\sqrt{x^2-b^2}}$.

9. Simplify $\frac{1}{1+\sqrt{2}+\sqrt{3}}+\frac{1}{1+\sqrt{2}-\sqrt{3}}+\frac{1}{1-\sqrt{2}+\sqrt{3}}$
$$+\frac{1}{\sqrt{2}+\sqrt{3}-1}.$$

10. If $y=\frac{1}{x+\sqrt{x^2-1}}$ shew that $2x=y+y^{-1}$.

CHAPTER XXII.

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

134. Definitions. When two algebraical expressions are connected by the sign $=$, the whole is called, according to circumstances, an *identity* or an *equation.*

An Identity *is merely the statement of the equality of two different forms of the same quantity, and is true for all values of the letters involved in it.* **Ex.** $a^2-b^2=(a+b)(a-b)$.

An Equation *is the statement of the equality of two different algebraical quantities; and is true only for some particular values of one or more of the letters contained in it.*

Thus the equation $x-2=4$, is true only when $x=6$.

The expressions $x-2$ and 4 are called the *sides* of the equation.

Equations are divided into classes according to the highest power of the unknown quantity or quantities involved in them.

Simple Equations are those which when reduced to their simplest form contain terms of only one dimension in the unknown quantities. Thus $x+y=5x+4$; and $ax+b=c$ are simple equations.

Quadratic Equations are those in which one term at least is of two dimensions in the unknown quantities. Thus $ax^2+bx+c=0$ and $x^2+xy+y^2=a^2$ are quadratic equations.

If the term of highest dimensions involved be of the third degree, the equation in which it occurs is called a *cubic* equation or an equation of the third degree, and so on.

135. Unknown Quantity. A letter to which a particular value or values must be given in order that the statement contained in an equation may be true is called an *unknown quantity.*

Root of an equation. The value or values of the unknown quantity or quantities which make both sides of the equation identically equal are called the *root* or *roots* of the equation.

136. Axioms* The following self-evident statements are of constant application in the solution of equations :—

(i) If equals be added to equals, the sums are equal.

(ii) If equals be taken from equals, the remainders are equal.

(iii) If equals be multiplied by equals, the products are equal.

(iv) If equals be divided by equals, the quotients are equal.

(v) The same powers of equals are equal.

(vi) The same roots of equals are equal.

Cor. 1. *A term may be taken from one side of an equation to the other, provided we change its sign.*

Thus if $x+p=q$, then $x=q-p$.

For by (ii), $x+p-p=q-p$, i.e., $x=q-p$.

Similarly, if $x-p=q$, then $x=q+p$.

For by (i), $x-p+p=q+p$, i.e., $x=q+p$.

Cor. 2. *When the same term preceded by like signs appears on both sides of an equation, we may omit or cancel it.*

Thus if $x+b=c+b$, then $x=c$.

For by (ii), $x+b-b=c+b-b$, i.e., $x=c$.

137. In this chapter we shall treat of simple equations involving one unknown quantity, under the following heads :—

I. **Equations not involving fractions.**

II. **Equations involving fractions (vulgar or decimal).**

III. **Equations containing radicals.**

General Rule.—"To solve a simple equation of one unknown quantity, we must clear the equation of fractions and radicals when any enter into it, bring all the unknown terms to one side, and all the known terms to the other; add all the terms on each side, and divide both sides by the co-efficient of the unknown quantity: the result is the required solution."

138. Equations not involving Fractions. We shall now work out some examples. The unknown quantity is always denoted by x.

* These axioms are also enunciated in the Chapter on Identities since operations with Identities are similar to those with equations.

Ex. 1. Solve $4x-15=2x+5$.

Since $4x-15=2x+5$, by transposition we have—

$4x-2x=15+5$. $\therefore 2x=20$. Divide both sides by 2.

$\therefore x=10$.

Ex. 2. Solve $3(x-4)-2(x+2)=4(x+5)$.

Removing brackets we have $3x-12-2x-4=4x+20$.

Transposing, $3x-2x-4x=20+12+4$ or $-3x=36$. $\therefore 3x=-36$. Multiply both sides by -1. Dividing by 3, $x=-12$.

Ex. 3. Solve $(x+2)(x+3)+(x+4)(x+5)=(2x+1)(x+4)$.

Multiplying out, we have—

$$x^2+5x+6+x^2+9x+20=2x^2+9x+4$$

$$\text{or } 2x^2+14x+26=2x^2+9x+4.$$

Cancelling $2x^2$ which we have on both sides,

$$14x+26=9x+4.$$

Transposing, $14x-9x=4-26$ or $5x=-22$.

Dividing by 5, $x=-4\frac{2}{5}$.

Ex. 4. Solve $(x+a)(x+b)=(x+c)(x+d)$.

Multiplying out, we have—

$$x^2+ax+bx+ab=x^2+cx+dx+cd.$$

Cancelling x^2 on both sides, and transposing,

$$ax+bx-cx-dx=cd-ab.$$

$$\therefore x(a+b-c-d)=cd-ab.$$

$$\therefore x=\frac{cd-ab}{a+b-c-d}.$$

EXERCISE 55.

Solve the following equations:—

1. $2x+3=x+17$. 2. $2x+7=10+x$. 3. $7x-6=6x-4$.
4. $6x+2=5x+4$. 5. $7x-3=5x+4$. 6. $8x-5=13-7x$.
7. $3x-15=2(x-4)$. 8. $9x-3-(4x+22)=0$.
9. $7x-4-(3x-11)=0$. 10. $21x-12=4(x-3)+3x+42$.
11. $34x+12=50+55x-185$. 12. $2x-5(1+x)-7=0$.
13. $ax+b(x+a)=c(x+b)$. 14. $(x-a)(x-b)=x(x-c)$.
15. $(x+1)(x+3)=(x+3)(x+4)$. 16. $(x-2)(x-6)=x(x-7)$.
17. $5(x+1)+6(x+2)=9(x+3)$. 18. $x-a=(b-a)x$.

19. $(x+1)(x+2)=(x-3)(x-4)$.

20. $x(x+2)+x(x+1)=(2x-1)(x+3)$.

21. $(x-1)(x-2)+(x-3)(x-4)=2(x-7)(x-8)+116$.

22. $(3x-1)^2+(4x+2)^2=(5x+2)^2+4$.

23. $(x+1)(x+2)(x+3)+x(x+7)(1-x)=0$. [+1.

24. $(x-1)(x-2)(x-3)+(x+1)(x+2)(x+3)=x(2x^2+21)$

25. $(a+b)(a-x)=(a-b)(b+x)$.

26. $(x-a)(x-b)=(x-c)(x-d)$.

27. $(2+x)(a-3)=4-2ax$. 28. $(m+n)(m-x)=m(n-x)$.

29. $2(x+a+b)=(x+a-b)+(x-a+b)$.

30. $a(x+b+c)+b(x+a+c)+c(x+a+b)=0$.

31. $x(x+a)+x(x+b)-2(x+a)(x+b)=0$.

32. $(x+a)^2=5a^2+(x-a)^2$.

33. $(3a-x)(a-b)-4b(a+x)+2ax=0$.

34. $(x+3a)(x-3b)+3(x-3a)(x+3b)=4(x-3a)(x-3b)$.

35. $(a+2x)(b+2x)=4(x+2a)(x+2b)$

36. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

37. $(2b+2c-x)^2+(2b-2c+x)^2-(2b-2d+x)^2$
$=(2b+2d-x)^2$.

38. $(x+a)^3+(x+b)^3+(x+c)^3=3(x+a)(x+b)(x+c)$.

39. $(6x+a)^2+(8x-b)^2=(10x+a)^2-10a^2$.

40. $(x-a)(x-b)+a^2+b^2=(x-a-b)^2$.

41. $(x-a)(2x-b)^2=(x-b)(2x-a)^2$.

42. $(x-a)(x-b)(x+2a+2b)=(x+2a)(x+2b)(x-a-b)$.

43. $(mx-a-b)(nx-b-d)=(mx-a-c)(nx-c-d)$.

44. $(x+a)(2x+b+c)^2=(x+b)(2x+a+c)^2$.

45. $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b)$.

46. $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3=0$.

47. $(b-c)^3(x-a)+(c-a)^3(x-b)+(a-b)^3(x-c)=0$.

48. $(b-c)(x+b^2)(x+c^2)+(c-a)(x+c^2)(x+a^2)+(a-b)$
$\times(x+a^2)(x+b^2)=0$.

49. $(b-c)^3(2x+b+c)+(c-a)^3(2x+c+a)+(a-b)^3$
$(2x+a+b)=0$.

50. $(x+a)(x+b)(x+c+d)-(x+c)(x+d)(x+a+b)=0$.

139. Equations involving Fractions.

If an equation contains fractions, it may be reduced to a form capable of solution, by multiplying each side of the equation by the L.C.M. of the denominators of the fractions.

Ex. 1. Solve $\frac{x}{2}-\frac{x}{3}=\frac{x}{5}+1.$

Multiply each side by 30, the L.C.M. of the denominators.

We have $15x-10x=6x+30$. $\quad\therefore\ 15x-10x-6x=30.$

$\therefore\quad -x=30. \qquad \therefore\ x=-30.$

Ex. 2. Solve $\frac{x-1}{2}+\frac{2x+3}{3}=\frac{6x+19}{8}.$

Multiply each side by 24, the L.C.M. of the denominators.

We have $12(x-1)+8(2x+3)=3(6x+19)$.

$\therefore\ 12x-12+16x+24=18x+57$

$\therefore\ 28x-18x=57+12-24. \qquad \therefore\ 10x=45.$

$\therefore\ x=4\frac{1}{2}.$

Ex. 3. Solve $\frac{\cdot05-\cdot01x}{\cdot1}-(\cdot03-\cdot02x)=\cdot03.$

Reducing the decimals to vulgar fractions, we find that the equation becomes $\dfrac{\frac{5}{100}-\frac{1}{100}x}{\frac{1}{10}}-\left(\frac{3}{100}-\frac{2}{100}x\right)=\frac{3}{100}$

$$\text{or}\quad \frac{5}{10}-\frac{1}{10}x-\frac{3}{100}+\frac{2}{100}x=\frac{3}{100}.$$

Multiply each side by 100, the L.C.M. of the denominators.

We have $50-10x-3+2x=3$.

$\therefore\ -8x=3+3-50=-44. \qquad \therefore\ x=\frac{11}{2}=5\frac{1}{2}.$

Ex. 4 Solve $\frac{3}{x-1}+\frac{2}{x+4}=\frac{7}{2x-2}+\frac{4}{3x+12}.$

The L.C.M. of the denominators is $6(x-1)(x+4)$.

Multiplying each side by $6(x-1)(x+4)$, we have—

$$18(x+4)+12(x-1)=21(x+4)+8(x-1).$$

$\therefore\ 18x+72+12x-12=21x+84+8x-8.$

$\therefore\ 18x+12x-21x-8x=84-8+12-72. \qquad \therefore\ x=16.$

Ex. 5. Solve $\frac{1}{x}+\frac{b}{x+a}=\frac{1+b}{x+b}$.

Multiplying each side by $x(x+a)(x+b)$, the L.C.M. of the denominators, we have $(x+a)(x+b)+bx(x+b)=x(x+a)(1+b)$.

$\therefore x^2+x(a+b)+ab+bx^2+b^2x=x^2+bx^2+ax+abx$.

Cancelling like terms on both sides, we get $bx+b^2x-abx=-ab$.

Dividing both sides by b, $x+bx-ax=-a$.

$\therefore x(1+b-a)=-a$. $\quad\therefore x=\frac{-a}{1+b-a}=\frac{a}{a-b-1}$.

In the following examples, the solution is facilitated by suitable transposition and combination of terms.

Ex. 6. Solve $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}$.

We have $\frac{1}{x-1}+\frac{2}{x-2}=\frac{2}{x-3}+\frac{1}{x-3}$.

By transposition, $\frac{1}{x-1}-\frac{1}{x-3}=\frac{2}{x-3}-\frac{2}{x-2}$,

$\therefore \frac{x-3-x+1}{(x-3)(x-1)}=\frac{2x-4-2x+6}{(x-3)(x-2)}$.

$\therefore \frac{-2}{(x-3)(x-1)}=\frac{2}{(x-3)(x-2)}$. $\quad\therefore \frac{-1}{x-1}=\frac{1}{x-2}$.

Multiply both sides by ~~$(x-1)(x-2)$~~. $(x-3)$

We have $-1(x-2)=x-1$. $\quad\therefore -x+2=x-1$.

$\therefore -2x=-3$. $\quad\therefore x=\frac{3}{2}=1\frac{1}{2}$.

Ex. 7. Solve $\frac{b-c}{x+a}+\frac{a-b}{x+b}=\frac{a-c}{x+c}$.

We have $\frac{b-c}{x+a}+\frac{a-b}{x+b}=\frac{a-b+b-c}{x+c}=\frac{a-b}{x+c}+\frac{b-c}{x+c}$.

By transposition, $(b-c)\left\{\frac{1}{x+a}-\frac{1}{x+c}\right\}=(a-b)\left\{\frac{1}{x+c}-\frac{1}{x+b}\right\}$. $\quad\therefore (b-c)\frac{c-a}{(x+a)(x+c)}=(a-b)\frac{b-c}{(x+c)(x+b)}$.

$\therefore \frac{c-a}{x+a}=\frac{a-b}{x+b}$. Multiply both sides by $(x+a)(x+b)$.

We have $(x+b)(c-a)=(a-b)(x+a)$.

$\therefore x(c-a)+b(c-a)=x(a-b)+a(a-b)$.

$\therefore x(c-a-a+b)=a(a-b)-b(c-a)=a^2-bc$.

$$\therefore \; x=\frac{a^2-bc}{b+c-2a}.$$

Ex 8. Solve $\frac{x-a^3}{a^2+a}+\frac{x-a^2}{a^2+1}+\frac{x-a}{a+1}=1+a+a^2$.

We have by transposition, $\frac{x-a^3}{a^2+a}-1+\frac{x-a^2}{a^2+1}-a+\frac{x-a}{a+1}-a^2=0$.

Simplifying, $\frac{x-a^3-a^2-a}{a^2+a}+\frac{x-a^2-a^3-a}{a^2+1}+\frac{x-a-a^3-a^2}{a+1}=0$.

$$\therefore \; (x-a^3-a^2-a)\left\{\frac{1}{a^2+a}+\frac{1}{a^2+1}+\frac{1}{a+1}\right\}=0.$$

Dividing both sides by $\frac{1}{a^2+a}+\frac{1}{a^2+1}+\frac{1}{a+1}$, we have $x-a^3-a^2-a=0$. $\therefore x=a^3+a^2+a$.

EXERCISE 56.

Solve the following equations :—

1. $\cdot03x+\cdot02=\cdot02x-\cdot06$.
2. $\cdot07(x-10)+\cdot54x=\cdot2(\cdot1-\cdot1x)-3(\cdot05-\cdot02x)$.
3. $\frac{\cdot01x}{\cdot02}-\frac{x}{30}=\frac{\cdot01x}{\cdot5}+1\cdot34$.
4. $(1-2x)(\cdot01-\cdot03x)-\cdot23=(\cdot6x+\cdot1)(\cdot1x-\cdot1)-\cdot03x$.
5. $3\cdot75x+\cdot5=2\cdot25x+8$. 6. $\frac{x}{\cdot5}-\frac{1}{\cdot05}+\frac{x}{\cdot005}-\frac{1}{\cdot0005}=0$.
7. $\cdot5x+\frac{\cdot45x-\cdot75}{\cdot6}=\frac{1\cdot2}{\cdot2}-\frac{\cdot3x-\cdot6}{\cdot9}$.
8. $\cdot5x+\frac{\cdot02x+\cdot07}{\cdot03}-\frac{x+2}{9}=9\cdot5$.

9. $\dfrac{4\cdot05}{9x} - \dfrac{\cdot3}{\cdot8 - 2x} = \dfrac{1\cdot8}{x} - \dfrac{3\cdot6}{2\cdot4 - 6x}$.

10. $\cdot011x + \dfrac{\cdot001x - \cdot125}{\cdot6} = \dfrac{5 - x}{\cdot03} - \cdot145$.

11. $\cdot65x + \dfrac{\cdot585x - \cdot975}{\cdot6} = \dfrac{1\cdot56}{\cdot2} - \dfrac{\cdot39x - \cdot78}{\cdot9}$.

12. $\cdot15x + \dfrac{\cdot135x - \cdot225}{\cdot6} = \dfrac{\cdot36}{\cdot2} - \dfrac{\cdot09x - \cdot18}{\cdot9}$.

13. $\dfrac{\cdot03x - \cdot01}{\cdot02} - \dfrac{\cdot02(x-1)}{\cdot03} = \dfrac{\cdot01x - \cdot03}{\cdot4} + \dfrac{\cdot21}{\cdot2}$.

14. $(\cdot1x + \cdot2)^2 + \cdot7(\cdot3x - \cdot1) = \cdot06(2x+4) + (\cdot1x - \cdot2)^2 - \cdot65$.

15. $\dfrac{3x}{4} + 12 = \dfrac{5x}{6} + 9$.

16. $\dfrac{x+1}{2} + \dfrac{x+2}{3} - 16 + \dfrac{x+3}{4} = 0$.

17. $\dfrac{x}{8} - \dfrac{x-1}{2\frac{1}{2}} = \dfrac{3x-4}{15} + \dfrac{x}{12}$.

18. $\dfrac{x-1}{2} + \dfrac{x-2}{3} + \dfrac{x-3}{4} = \dfrac{5x-1}{6}$.

19. $\dfrac{2x-6}{5} - \dfrac{x-4}{9} - \dfrac{3x}{13} = 0$.

20. $\dfrac{4x-17}{9} - \dfrac{3\frac{2}{3} - 22x}{33} = x - \dfrac{6}{x}\left(1 - \dfrac{x^2}{54}\right)$.

21. $\dfrac{x+10}{3} - \frac{2}{5}(3x-4) + \dfrac{(3x-2)(2x-3)}{6} = x^2 - \dfrac{2}{15}$.

22. $\dfrac{11x-13}{25} + \dfrac{19x+3}{7} - \dfrac{5x - 25\frac{1}{4}}{4} = 28\frac{3}{7} - \dfrac{17x+4}{21}$.

23. $\dfrac{10x+17}{18} - \dfrac{12x+1}{11x-8} = \dfrac{5x-4}{9}$.

24. $\dfrac{1}{x-1} - \dfrac{2}{x-7} = \dfrac{1}{7(x-1)}$.

25. $\dfrac{1}{x-3} + \dfrac{1}{x-6} = \dfrac{2}{x-9}$.

26. $\dfrac{2}{x-4} - \dfrac{5}{x-2} + \dfrac{3}{x-6} = 0$.

27. $\dfrac{x}{a} + \dfrac{x}{b-a} = \dfrac{a}{b+a}$.

28. $\dfrac{1}{x-3} - \dfrac{1}{x-4} = \dfrac{3}{3x-13} - \dfrac{3}{3x-16}$.

29. $\dfrac{1}{x-1}-\dfrac{3}{x+6}+\dfrac{3}{x-3}-\dfrac{1}{x-4}=0.$

30. $\dfrac{3}{x-2}+\dfrac{5}{x-6}=\dfrac{8}{x+3}.$ 31. $\dfrac{8}{2x-1}+\dfrac{9}{3x-1}=\dfrac{7}{x+1}.$

32. $\dfrac{2}{2x-5}+\dfrac{1}{x-3}=\dfrac{6}{3x-1}.$ 33. $\dfrac{3}{4x+1}+\dfrac{4}{4x+5}=\dfrac{7}{4x+3}.$

34. $\dfrac{1}{x+a}+\dfrac{1}{x+c}=\dfrac{2}{x+b}.$ 35. $\dfrac{3}{1-3x}+\dfrac{5}{1-5x}+\dfrac{4}{2x-1}=0.$

36. $\dfrac{21}{3x-7}-\dfrac{5}{x+3}=\dfrac{2}{x-5}.$ 37. $\dfrac{1}{x+1}-\dfrac{2}{x+2}+\dfrac{1}{x+4}=0.$

38. $\dfrac{a-c}{x+2b}+\dfrac{b-c}{x+2a}=\dfrac{a+b-2c}{a+b+c}.$

39. $\dfrac{x-6ab}{3a+2b}+\dfrac{x-3ac}{3a+c}+\dfrac{x-2bc}{2b+c}=3a+2b+c.$

40. $\dfrac{x-ab}{a+b}+\dfrac{x-a}{a+1}+\dfrac{x-b}{1+b}=1+b+a.$

41. $\dfrac{x-ab}{a+b}+\dfrac{x-bc}{b+c}+\dfrac{x-ca}{c+a}=a+b+c.$

42. $\dfrac{x-a^2}{b^2+c^2}+\dfrac{x-b^2}{c^2+a^2}+\dfrac{x-c^2}{a^2+b^2}=3.$

43. $\dfrac{x+a}{b+c+d}+\dfrac{x+b}{c+d+a}+\dfrac{x+c}{d+a+b}+\dfrac{x+d}{a+b+c}+4=0.$

44. $\dfrac{1}{x+6a}+\dfrac{2}{x-3a}+\dfrac{3}{x+2a}=\dfrac{6}{x+a}.$

45. $\dfrac{1}{x-7}+\dfrac{1}{x-13}=\dfrac{1}{x-9}+\dfrac{1}{x-11}.$

46. $\dfrac{1}{x-6a}+\dfrac{2}{x+3a}+\dfrac{3}{x-2a}=\dfrac{6}{x-a}.$

47. $\dfrac{b+c}{x-a}=\dfrac{b}{x-c}+\dfrac{c}{x-b}.$ 48. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}.$

49. $\dfrac{2}{2x-3}+\dfrac{1}{x-2}=\dfrac{6}{3x+2}.$ 50. $\dfrac{a+b}{x-c}=\dfrac{a}{x-b}+\dfrac{b}{x-a}.$

51. $\dfrac{1}{x-2}-\dfrac{1}{x-4}=\dfrac{1}{x-6}-\dfrac{1}{x-8}$.

52. $\dfrac{1}{x-9}-\dfrac{1}{x-11}=\dfrac{1}{x-15}-\dfrac{1}{x-17}$.

53. $\dfrac{1}{x-a}-\dfrac{1}{x-a+c}=\dfrac{1}{x-b-c}-\dfrac{1}{x-b}$.

54. $\dfrac{x-a}{a-b}-\dfrac{x+a}{a+b}=\dfrac{2ax}{a^2-b^2}$. 55. $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{x+a}=m+n$.

140. Equations involving Fractions—Special Methods.

A. *Write the fractions in mixed form, by dividing the numerator by the denominator.*

Ex. 1. Solve $\dfrac{x-1}{x-2}=\dfrac{x-2}{x-5}$.

Dividing out each fraction, we have $1+\dfrac{1}{x-2}=1+\dfrac{3}{x-5}$.

$\therefore \quad \dfrac{1}{x-2}=\dfrac{3}{x-5}, \quad \therefore \; x-5=3(x-2)=3x-6,$

$\therefore \quad -2x=-1, \quad \therefore \; x=\frac{1}{2}.$

Ex. 2. Solve $\dfrac{x^3+7x^2+24x+30}{x^2+5x+13}=\dfrac{2x^3+11x^2+36x+45}{2x^2+7x+20}$.

Dividing out each fraction, we have—

$$x+2+\frac{x+4}{x^2+5x+13}=x+2+\frac{2x+5}{2x^2+7x+20}.$$

$$\therefore \frac{x+4}{x^2+5x+13}=\frac{2x+5}{2x^2+7x+20}, \quad \therefore \frac{x^2+5x+13}{x+4}=\frac{2x^2+7x+20}{2x+5}.$$

Dividing the numerator by the denominator, we have—

$$x+1+\frac{9}{x+4}=x+1+\frac{15}{2x+5}, \quad \therefore \; \frac{9}{x+4}=\frac{15}{2x+5},$$

$$\therefore \frac{3}{x+4}=\frac{5}{2x+5}, \quad \therefore 6x+15=5x+20, \quad \therefore x=5.$$

Ex 3. Solve $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$.

Dividing out each fraction, we have—

$$1+\frac{1}{x-2}-\left(1+\frac{1}{x-3}\right)=1+\frac{1}{x-6}-\left(1+\frac{1}{x-7}\right).$$

$$\therefore \frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-6}-\frac{1}{x-7}.$$

$$\therefore \frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}.$$ Multiply both sides by -1,

$$\frac{1}{(x-2)(x-3)}=\frac{1}{(x-6)(x-7)}. \qquad \therefore \frac{x-6}{x-2}=\frac{x-3}{x-7}.$$

Dividing out each fraction, we have $1-\frac{4}{x-2}=1+\frac{4}{x-7}$.

$$\therefore \frac{-1}{x-2}=\frac{1}{x-7}, \quad \therefore -(x-7)=x-2, \quad \therefore -2x=-9, \quad \therefore x=4\tfrac{1}{2}.$$

EXERCISE 57.

Solve the following equations :—

1. $\frac{x+4}{x+8}=\frac{x+8}{x+14}$. 2. $\frac{x-7}{x-5}=\frac{2x-7}{2x-5}$. 3. $\frac{7x-4}{x-1}=\frac{7x-26}{x-3}$.

4. $\frac{2x-7}{2x-3}=\frac{x-2}{x+2}$. 5. $\frac{x-4}{x-1}+\frac{x-2}{x-4}=2$.

6. $\frac{x+15}{x+10}+\frac{x+3}{x+6}=2$. 7. $\frac{x-1}{x-3}+\frac{x-5}{x-1}=2$.

8. $\frac{3(x+5)}{x+6}+\frac{4(x+6)}{x+5}=7$. 9. $\frac{2x+15}{2x+9}+\frac{2x-1}{2x-3}=2$.

10. $\frac{5x^2+x-3}{5x-4}=\frac{7x^2-3x-9}{7x-10}$. 11. $\frac{3x}{x+1}+\frac{3x-2}{3x-5}=4$.

12. $\frac{x+a}{x-a}=\frac{x+b}{x-b}$. 13. $\frac{3x+5}{3x-5}+\frac{2x+4}{x-2}=3$.

14. $\frac{x^2+15x+56}{x+8}=\frac{2x^2+23x+63}{x+7}$.

15. $\frac{2x^3+5x^2+7x+5}{2x^2+3x+2}=\frac{x^3+9x^2+11x+10}{x^2+8x+2}$.

49. $\frac{4x^3+6x^2+11x+7}{4x^2+2x+5}=\frac{2x^3+6x^2+11x+9}{2x^2+4x+5}$.

17. $\dfrac{2x^3+15x^2+18x-34}{2x^2+9x-11}=\dfrac{x^3+4x^2-20x-76}{x^2+x-24}.$

18. $\dfrac{x^2+2x-2}{x-1}+\dfrac{x^2-2x-2}{x+1}=\dfrac{2x^2-6x+2}{x-3}.$

19. $\dfrac{x^4+10x^3+38x^2+65x+43}{x^3+6x^2+13x+10}=\dfrac{5x^3+46x^2+147x+163}{5x^2+26x+38}.$

20. $\dfrac{x+4a+b}{x+a+b}+\dfrac{4x+a+2b}{x+a-b}=5$

21. $\dfrac{x+2a}{x-2b}=\left(\dfrac{x+a}{x-b}\right)^2.$ 22. $\dfrac{2x-3}{x+1}=\dfrac{4x+5}{4x+4}+\dfrac{3x+3}{3x+1}.$

23. $\dfrac{x-7}{x-3}+\dfrac{x-2}{x-9}+\dfrac{x-4}{x-1}=3$ 24. $\dfrac{x-5}{x-4}-\dfrac{x-6}{x-5}=\dfrac{x-3}{x-4}-\dfrac{x-2}{x-3}.$

25. $\dfrac{x-4}{x-5}+\dfrac{x-6}{x-7}=\dfrac{x-2}{x-3}+\dfrac{x-8}{x-9}$

26. $\dfrac{x+3}{x+1}+\dfrac{x-6}{x-4}=\dfrac{x-5}{x-3}+\dfrac{x+4}{x+2}$

27. $\dfrac{x+6}{x+7}+\dfrac{x+11}{x+12}=\dfrac{x+7}{x+8}+\dfrac{x+10}{x+11}$ 28. $\dfrac{x+a+b}{x-a-b}=\dfrac{x-a-b}{x+a+b}$

29. $\dfrac{x+4a-2b}{x+a-2b}+\dfrac{2x-a-2b}{x+a-4b}=3.$

30. $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}=\dfrac{x-a-b}{x-c-d}.$ 31. $\dfrac{x+a}{x+b}=\left(\dfrac{2x+a+c}{2x+b+c}\right)^2.$

32. $\dfrac{x-a}{x-a-1}-\dfrac{x-a-1}{x-a-2}=\dfrac{x-b}{x-b-1}-\dfrac{x-b-1}{x-b-2}.$

33. $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}.$ 34. $\dfrac{3+x}{3-x}-\dfrac{2+x}{2-x}-\dfrac{1+x}{1-x}=1$

35. $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x+a-2b}.$

36. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+17}{x+4}=\dfrac{x^2+4x+5}{x+2}+\dfrac{x^2+6x+10}{x+3}.$

37. $\dfrac{x^2-2x+2}{x-1}+\dfrac{x^2-8x+20}{x-4}=\dfrac{x^2-4x+6}{x-2}+\dfrac{x^2-6x+12}{x-3}.$

38. $\dfrac{x-4}{x-5}+\dfrac{x-8}{x-9}=\dfrac{x-7}{x-8}+\dfrac{x-5}{x-6}.$

39. $(x+2a)(x+2b)(x-a-b)=(x-a)(x-b)(x+2a+2b).$

40. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}.$

41. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{11x+18}{11x-18}.$

42. $\dfrac{x-5}{x+5}-\dfrac{x+5}{x-5}+\dfrac{x^2}{x^2-25}=1.$

43. $\dfrac{2x-1}{2x-3}+\dfrac{2x-7}{2x-5}+\dfrac{3x-13}{3x-4}+\dfrac{3x+4}{3x-5}=4.$

44. $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}.$

45. $\dfrac{x+a}{x-a}+\dfrac{x+4a}{x-4a}=\dfrac{x+3a}{x-3a}+\dfrac{x+2a}{x-2a}.$

46. $\dfrac{4x-3}{4x-5}-\dfrac{4x-7}{4x-9}=\dfrac{4x-11}{4x-13}-\dfrac{4x-15}{4x-17}.$

47. $\dfrac{7x-55}{x-8}+\dfrac{2x-17}{x-9}=\dfrac{6x-71}{x-12}+\dfrac{3x-14}{x-5}.$

48. $\dfrac{x^2+3x+3}{x+2}+\dfrac{x^2-15}{x-4}=\dfrac{x^2+7x+11}{x+5}+\dfrac{x^2-4x-20}{x-7}.$

49. $\dfrac{x+7a}{x+6a}+\dfrac{x-a}{x-3a}+\dfrac{a-x}{x+2a}=\dfrac{x+7a}{x+a}.$

50. $\dfrac{x^2-(a-b)^2+c^2}{x-a+b}-\dfrac{(x+a-b)^2+c^2}{x+a-b}=2\left(\dfrac{c^2}{a+b}-\dfrac{c^2}{x+a}\right).$

B. *When an equation consists of two fractions only, its solution may be simplified by the application of one or other of the following results*:—

If $\dfrac{a}{b}=\dfrac{c}{d}$, then (1) $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{a+c}{b+d}=\dfrac{a-c}{b-d}.$

(2) $\dfrac{a+b}{b}=\dfrac{c+d}{d}$; $\dfrac{a-b}{b}=\dfrac{c-d}{d}.$ (3) $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}.$

Ex. 1. Solve $\dfrac{mx-a-b}{nx-c-d}=\dfrac{mx-a-c}{nx-b-d}.$

$\therefore \dfrac{mx-a-b}{nx-c-d}=\dfrac{mx-a-b-mx+a+c}{nx-c-d-nx+b+d}$ . . . by (1).

$=\dfrac{c-b}{b-c}=-1.$ $\therefore mx-a-b=-nx+c+d.$

$\therefore\ x(m+n)=a+b+c+d.\quad \therefore\ x=\dfrac{a+b+c+d}{m+n}.$

Ex 2. Solve $x^2\dfrac{x^2+3a^2}{c^2+3x^2}=ac\dfrac{a^2+3x^2}{x^2+3c^2}.$

$$\therefore\ \frac{x(x^2+3a^2)}{c(c^2+3x^2)}=\frac{a(a^2+3x^2)}{x(x^2+3c^2)}.\quad \therefore\ \frac{x^3+3a^2x}{a^3+3ax^2}=\frac{c^3+3cx^2}{x^3+3xc^2}.$$

$$\therefore\ \frac{x^3+3a^2x+a^3+3ax^2}{x^3+3a^2x-a^3-3ax^2}=\frac{c^3+3cx^2+x^3+3xc^2}{c^3+3cx^2-x^3-3xc^2}\ \text{by (3)}.$$

$$\therefore\ \left(\frac{x+a}{x-a}\right)^3=\left(\frac{c+x}{c-x}\right)^3.\quad \therefore\ \frac{x+a}{x-a}=\frac{c+x}{c-x}.$$

Again by (3), $\dfrac{x+a+x-a}{x+a-x+a}=\dfrac{c+x+c-x}{c+x-c+x}.$

$$\therefore\ \frac{x}{a}=\frac{c}{x}.\quad \therefore\ x^2=ac.\quad \therefore\ x=\pm\sqrt{ac}.$$

Ex. 3. Solve $(x-1)(x-5)(x-7)(x-9)=(x-2)(x-4)$
$\times(x-6)(x-10).$

We may write it thus: $\dfrac{(x-1)(x-5)}{(x-6)(x-10)}=\dfrac{(x-2)(x-4)}{(x-7)(x-9)}.$

$$\therefore\ \frac{x^2-6x+5}{x^2-16x+60}=\frac{x^2-6x+8}{x^2-16x+63}.$$

$$\therefore\ \text{by (1)},\ \frac{x^2-6x+5}{x^2-16x+60}=\frac{x^2-6x+5-x^2+6x-8}{x^2-16x+60-x^2+16x-63}$$

$$=\frac{-3}{-3}=1.$$

$\therefore\ x^2-6x+5=x^2-16x+60.\qquad \therefore\ 10x=55.\qquad \therefore\ x=5\tfrac{1}{2}.$

Ex 4. Solve $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b).$

$$\therefore\ \frac{(x-a)^3}{(x-b)^3}=\frac{x-2a+b}{x+a-2b}.$$

$$\therefore\ \text{by (3)},\ \frac{(x-a)^3+(x-b)^3}{(x-a)^3-(x-b)^3}=\frac{x-2a+b+x+a-2b}{x-2a+b-x-a+2b}.$$

$$\therefore\ \frac{(2x-a-b)\{x^2-(a+b)x+a^2-ab+b^2\}}{(b-a)\{3x^2-3(a+b)x+a^2+ab+b^2\}}=\frac{2x-a-b}{3(b-a)}.$$

$$\therefore\ 3(2x-a-b)\{x^2-(a+b)x+a^2-ab+b^2\}$$
$$=(2x-a-b)\{3x^2-3(a+b)x+a^2+b^2+ab\}.$$

$\therefore (2x-a-b)\{2a^2-4ab+2b^2\}=0.$

$\therefore 2x-a-b=0. \quad \therefore 2x=a+b. \quad \therefore x=\dfrac{a+b}{2}.$

EXERCISE 58.

Solve the following equations :—

1. $\dfrac{x-1}{x+1}=\dfrac{1-a}{1+a}.$ 2. $\dfrac{x-a}{x-b}=\dfrac{x-m+n}{x+m-n}.$

3. $\dfrac{ax-b-c}{px-q-r}=\dfrac{ax-b-q}{px-c-r}.$ 4. $\dfrac{mx+n}{mx-n}=\dfrac{b+c-a}{c+a-b}.$

5. $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}.$ 6. $\dfrac{x+a}{x-a}=\dfrac{(a-c)(b+d)}{(a+c)(b-d)}.$

7. $(x-1)(2x-3)^2=(x-3)(2x-1)^2.$

8. $\dfrac{x-b-a}{x+2a+2b}=\dfrac{(x-a)(x-b)}{(x+2a)(x+2b)}.$

9. $x^2 \cdot \dfrac{x^2+3a^4}{c^4+3x^2}=a^2c^2 \cdot \dfrac{a^4+3x^2}{x^2+3c^4}.$

10. $\dfrac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)}=ax.$

11. $(x+1)(2x+5)^2=4(x+2)^3.$

12. $(x-a)(2x-b)^2=(x-b)(2x-a)^2.$

13. $(x+1)(x+2)(x+3)=(x+4)(x+5)(x-3).$

14. $(x+5)(x+6)^2(x+9)=(x+4)(x+7)^2(x+8).$

15. $(x+1)(x+6)^2=(x+2)(x+4)(x+7).$

16. $(x+2a+b)(x+a+2b)=(x+a+b)^2.$

17. $(x-1)(x-2)(x+6)=(x+2)(x-3)(x+4).$

18. $(x-1)(x-2)^2(x-5)=x(x-3)^2(x-4).$

141. Equations involving Radicals.

Ex. 1. Solve $\sqrt{x+4}+\sqrt{x+11}=7.$

Transposing, we have, $\sqrt{x+4}=7-\sqrt{x+11}.$

Squaring, $x+4=49+x+11-14\sqrt{x+11}.$

$\therefore 14\sqrt{x+11}=56. \quad \therefore \sqrt{x+11}=4.$ Squaring, $x+11=16$

$\therefore x=5.$

Or thus: $\sqrt{x+4}+\sqrt{x+11}=7$ . . . . (A).

Identically, $(x+4)-(x+11)=-7$ . . . . (B).

Dividing (B) by (A), $\sqrt{x+4}-\sqrt{x+11}=-1$. . . . (C).

Adding (A) and (C), $2\sqrt{x+4}=6$. $\therefore\ \sqrt{x+4}=3$.

Squaring, $x+4=9$. $\therefore\ x=5$.

Ex. 2. Solve $\sqrt{cx-a}-\sqrt{cx+b}+\sqrt{dx+a}-\sqrt{dx-b}=0$.

Writing it thus: $\sqrt{cx-a}-\sqrt{cx+b}=\sqrt{dx-b}-\sqrt{dx+a}$ . . . (A).

Identically, $(cx-a)-(cx+b)=(dx-b)-(dx+a)$ . . . . (B).

Dividing (B) by (A), $\sqrt{cx-a}+\sqrt{cx+b}=\sqrt{dx-b}+\sqrt{dx+a}$ (C).

Adding (A) and (C), $2\sqrt{cx-a}=2\sqrt{dx-b}$.

$\therefore \sqrt{cx-a}=\sqrt{dx-b}$. Squaring, $cx-a=dx-b$.

$\therefore\ x(c-d)=a-b$. $\therefore\ x=\dfrac{a-b}{c-d}$.

Ex. 3. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$.

Applying (3) of Art (140 B), $\dfrac{\sqrt{a+x}+\sqrt{a-x}+\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}-\sqrt{a+x}+\sqrt{a-x}}$

$=\dfrac{b+1}{b-1}$.

$\therefore\ \dfrac{\sqrt{a+x}}{\sqrt{a-x}}=\dfrac{b+1}{b-1}$. Squaring, $\dfrac{a+x}{a-x}=\dfrac{(b+1)^2}{(b-1)^2}$.

Applying the same formula, $\dfrac{a+x+a-x}{a+x-a+x}=\dfrac{(b+1)^2+(b-1)^2}{(b+1)^2-(b-1)^2}$.

$\therefore\ \dfrac{a}{x}=\dfrac{b^2+1}{2b}$. $\therefore\ x=\dfrac{2ab}{b^2+1}$.

Ex. 4. Solve $\sqrt{\dfrac{1+x}{1-x}}+\sqrt{\dfrac{1-x}{1+x}}=2$.

Putting a for $\sqrt[4]{\dfrac{1+x}{1-x}}$ and $\dfrac{1}{a}$ for $\sqrt[4]{\dfrac{1-x}{1+x}}$, we have—

$a^2+\dfrac{1}{a^2}=2$. $\therefore\ a^2-2+\dfrac{1}{a^2}=0$. $\therefore\ \left(a-\dfrac{1}{a}\right)^2=0$

$\therefore\ a-\dfrac{1}{a}=0$. $\therefore\ a=\dfrac{1}{a}$. $\therefore\ a^2=1$.

$\therefore \left\{\sqrt[4]{\dfrac{1+x}{1-x}}\right\}^2=1.\quad \therefore \sqrt{\dfrac{1+x}{1-x}}=1.\quad \therefore \dfrac{1+x}{1-x}=1.$

$\therefore\ 1+x=1-x.\quad \therefore\ 2x=0.\quad \therefore\ x=0.$

Ex 5 Solve $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b.$

Cubing both sides,

$a+x+a-x+3\sqrt[3]{a+x}\sqrt[3]{a-x}(\sqrt[3]{a+x}+\sqrt[3]{a-x})=b^3.$

$\therefore\ 2a+3\sqrt[3]{a^2-x^2}(\sqrt[3]{a+x}+\sqrt[3]{a-x})=b^3.$

But $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b.\quad \therefore\ 2a+3\sqrt[3]{a^2-x^2}.\ (b)=b^3.$

$\therefore\ 3b\sqrt[3]{a^2-x^2}=b^3-2a.\quad \therefore\ \sqrt[3]{a^2-x^2}=\dfrac{b^3-2a}{3b}.$

Cubing both sides, $a^2-x^2=\left(\dfrac{b^3-2a}{3b}\right)^3.$

$\therefore x^2=a^2-\left(\dfrac{b^3-2a}{3b}\right)^3.\ \therefore x=\pm\sqrt{\left\{a^2-\left(\dfrac{b^3-2a}{3b}\right)^3\right\}}.$

Ex. 6. Solve $\dfrac{x^2+16a^2}{x+4a+\sqrt{8ax}}-\dfrac{x^2+64a^2}{x+8a+\sqrt{16ax}}=-2a\sqrt{2}.$

$x^2+16a^2=(x+4a)^2-8ax=(x+4a+\sqrt{8ax})(x+4a-\sqrt{8ax})$

and $x^2+64a^2=(x+8a)^2-16ax=(x+8a+\sqrt{16ax})(x+8a-\sqrt{16ax}).$

$\therefore$ the equation becomes $x+4a-\sqrt{8ax}-(x+8a-\sqrt{16ax})=-2a\sqrt{2}.$

$\therefore\ \sqrt{16ax}-\sqrt{8ax}=4a-2a\sqrt{2}.\ \therefore\sqrt{ax}(4-2\sqrt{2})=a(4-2\sqrt{2}).$

$\therefore\ \sqrt{ax}=a.\quad \therefore\ ax=a^2.\quad \therefore\ x=a.$

Ex. 7. Solve $\sqrt[2pq]{x^{p+q}}-\dfrac{1}{2c}(\sqrt[p]{x}+\sqrt[q]{x})=0.$

$\therefore\ x^{\frac{p+q}{2pq}}=\dfrac{1}{2c}\left(x^{\frac{1}{p}}+x^{\frac{1}{q}}\right).\quad \therefore\ \dfrac{x^{\frac{1}{p}}+x^{\frac{1}{q}}}{2x^{\frac{p+q}{2pq}}}=c.$

$\therefore\ \dfrac{x^{\frac{1}{p}}+x^{\frac{1}{q}}+2x^{\frac{p+q}{2pq}}}{x^{\frac{1}{p}}+x^{\frac{1}{q}}-2x^{\frac{p+q}{2pq}}}=\dfrac{c+1}{c-1}.\quad \therefore\ \left(\dfrac{x^{\frac{1}{2p}}+x^{\frac{1}{2q}}}{x^{\frac{1}{2p}}-x^{\frac{1}{2q}}}\right)^2=\dfrac{c+1}{c-1}.$

$$\therefore \frac{x^{\frac{1}{2p}} + x^{\frac{1}{2q}}}{x^{\frac{1}{2p}} - x^{\frac{1}{2q}}} = \frac{\sqrt{c+1}}{\sqrt{c-1}}. \quad \therefore \frac{x^{\frac{1}{2p}}}{x^{\frac{1}{2q}}} = \frac{\sqrt{c+1}+\sqrt{c-1}}{\sqrt{c+1}-\sqrt{c-1}}.$$

$$\therefore x^{\frac{1}{2p}-\frac{1}{2q}} \text{ or } x^{\frac{q-p}{2pq}} = \frac{\sqrt{c+1}+\sqrt{c-1}}{\sqrt{c+1}-\sqrt{c-1}}.$$

$$\therefore x = \left(\frac{\sqrt{c+1}+\sqrt{c-1}}{\sqrt{c+1}-\sqrt{c-1}}\right)^{\frac{2pq}{q-p}}.$$

Ex 8. Solve $\sqrt[3]{x+a}+\sqrt[3]{x+b}+\sqrt[3]{x+c}=0$.

If $p+q+r=0$, then $p^3+q^3+r^3=3pqr$

$\therefore x+a+x+b+x+c=3\sqrt[3]{(x+a)(x+b)(x+c)}$

$\therefore (3x+a+b+c)^3=27(x+a)(x+b)(x+c)$

$\therefore 27x^3+27x^2(a+b+c)+9x(a+b+c)^2+(a+b+c)^3$
$\qquad =27x^3+27x^2(a+b+c)+27x(ab+bc+ca)+27abc.$

$\therefore 9x\{(a+b+c)^2-3(ab+ac+bc)\}=27abc-(a+b+c)^3.$

$$\therefore x=\frac{27abc-(a+b+c)^3}{9(a+b+c)^2-27(ab+bc+ca)}$$
$$=\frac{27abc-(a+b+c)^3}{9(a^2+b^2+c^2-ab-ac-bc)}.$$

Ex. 9. Solve $\sqrt{a+x}+\sqrt{a-x}=\dfrac{2x}{\sqrt{a+\sqrt{a^2+x^2}}}$.

Writing it thus $\sqrt{a+x}+\sqrt{a-x}=\dfrac{a+x-(a-x)}{\sqrt{a+\sqrt{a^2+x^2}}}$,

$\sqrt{a+x}+\sqrt{a-x}$ is a factor of each side.

Dividing each side by it, we have $1=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+\sqrt{a^2+x^2}}}$.

$\therefore \sqrt{a+\sqrt{a^2+x^2}}=\sqrt{a+x}-\sqrt{a-x}$

Squaring, $a+\sqrt{a^2+x^2}=a+x+a-x-2\sqrt{a^2-x^2}$

$\therefore \sqrt{a^2+x^2}-a=-2\sqrt{a^2-x^2}$

Squaring, $a^2+x^2+a^2-2a\sqrt{a^2+x^2}=4(a^2-x^2)$.

$\therefore 5x^2-2a^2=2a\sqrt{a^2+x^2}$.

$\therefore 25x^4+4a^4-20a^2x^2=4a^4+4a^2x^2 \quad \therefore 25x^4-24a^2x^2=0.$

$\therefore 25x^2=24a^2 \quad \therefore x^2=\frac{24}{25}a^2. \quad \therefore x=\pm\frac{a}{5}\sqrt{24}=\pm\frac{2a}{5}\sqrt{6}.$

EXERCISE 59.

Solve the following equations :—

1. $\sqrt{4x}+\sqrt{4x-7}=7$.
2. $\sqrt{(x+14)}+\sqrt{(x-14)}=14$.
3. $\sqrt{x+11}+\sqrt{x-9}=10$.
4. $\sqrt{(9x+4)}+\sqrt{(9x-1)}=3$.
5. $\sqrt{(x+4ab)}=2a-\sqrt{x}$.
6. $\sqrt{x-a}+\sqrt{x-b}=\sqrt{a-b}$.
7. $\sqrt{(x+3a)}-\sqrt{(x-2a)}=\sqrt{a}$.
8. $\sqrt{x+2}+\sqrt{x-3}=5$
9. $\sqrt{x+4a}-\sqrt{x+2a}=2\sqrt{x-a}$.
10. $\sqrt{x-4}+\sqrt{x+5}=\sqrt{4x+1}$.
11. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$.
12. $x+\sqrt{2ax+x^2}=a$.
13. $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$.
14. $a+x+\sqrt{2ax+x^2}=b$.
15. $\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}=\sqrt[3]{b}$.
16. $\dfrac{x-1}{\sqrt{x}+1}=4+\dfrac{\sqrt{x}-1}{2}$.
17. $\sqrt{x}+\sqrt{x+a}=\dfrac{2a}{\sqrt{x+a}}$.
18. $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$.
19. $\dfrac{5x-9}{\sqrt{5x}+3}-1=\dfrac{\sqrt{5x}-3}{2}$.
20. $\dfrac{ax-b^2}{\sqrt{ax}+b}+c=\dfrac{\sqrt{ax}-b}{c}$.
21. $\sqrt{5x-14}-\sqrt{5x-21}=1$.
22. $\sqrt{x^2+3}+\sqrt{x^2+8}=5$.
23. $\sqrt{\dfrac{x+\sqrt{b}}{\sqrt{b}-x}}+\sqrt{\dfrac{\sqrt{b}-x}{x+\sqrt{b}}}=2$.
24. $\sqrt{\dfrac{ax+\sqrt{c}}{\sqrt{c}-ax}}+\sqrt{\dfrac{\sqrt{c}-ax}{ax+\sqrt{c}}}=2$.
25. $\sqrt{x+p^2}+\sqrt{x+q^2}=\sqrt{2x-p^2}+\sqrt{2x-q^2}$.
26. $\sqrt{x+a-c}+\sqrt{x+b-c}=\sqrt{c-a}+\sqrt{c-b}$.
27. $\sqrt{x+7}-\sqrt{x-3}=\sqrt{3x-14}-\sqrt{3x-24}$.
28. $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34}$.
29. $\sqrt{x-2}+\sqrt{x-7}=\sqrt{3x-10}-\sqrt{3x-15}$.
30. $\sqrt{5x-4}-\sqrt{4x-3}=-\sqrt{2x-1}+\sqrt{3x-2}$.
31. $\sqrt{ax+b}+\sqrt{ax+c}=\sqrt{dx+b}+\sqrt{dx+c}$.
32. $\sqrt{(x+2+\sqrt{x^2+2})}+\sqrt{(x-2+\sqrt{x^2+2})}=2$.
33. $\sqrt{x+3}+\sqrt{x-6}=3(1+\sqrt{2})$.
34. $\sqrt{a-x}+\sqrt{b-x}-\sqrt{c+x}=\sqrt{a-b+c+x}$.

35. $\frac{x^2-a}{x-\sqrt{a}}+\frac{x^2-a}{x+\sqrt{a}}=x-\frac{\sqrt{a}}{2}.$

36. $\frac{1-x}{1+\sqrt{a}}+\frac{1+x}{1-\sqrt{a}}=\frac{2}{1-a^2}.$

37. $\frac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4.$

38. $\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}=1.$

39. $\frac{a+\sqrt{2ax-a^2}}{a-\sqrt{2ax-a^2}}=a$

40. $\frac{\sqrt{36x+1}+\sqrt{36x}}{\sqrt{36x+1}-\sqrt{36x}}=9$

41. $\frac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\frac{a}{x-a}$

42. $\frac{\sqrt{4x+5}+\sqrt{x}}{\sqrt{4x+5}-\sqrt{x}}=2.$

43. $\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\frac{m+n}{m-n}.$

44. $\frac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=b$

45. $\frac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2.$

46. $\frac{x^3+3x}{3x^2+1}=p.$

47. $\frac{x^4+6x^2+1}{4x(x^2+1)}=a.$

48. $\frac{\sqrt{ax+b}+\sqrt{ax-b}}{\sqrt{ax+b}-\sqrt{ax-b}}=\left(\frac{p+1}{p-1}\right)^2.$

49. $\frac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}}=a$

50. $\frac{1+x+x^2}{1-x+x^2}=\frac{62}{63}\times\frac{1+x}{1-x}.$

51. $\frac{x-9}{\sqrt{x-5}-2}+\frac{x-20}{\sqrt{x-4}+4}=\frac{4x-7}{\sqrt{4x-3}+2}$

52. $\frac{1}{x+\sqrt{x^2-a^2}}+\frac{1}{x-\sqrt{x^2-a^2}}=4.$

53. $\frac{1}{\sqrt{a+x}+\sqrt{a}}+\frac{1}{\sqrt{a+x}-\sqrt{a}}=\frac{\sqrt{a}}{x}.$

54. $\sqrt[3]{x+a}-\sqrt[3]{x-a}=2\sqrt[3]{a}$

55. $x+a-b+3\sqrt[3]{abx}=0.$

56. $b+b^2+3b\sqrt[3]{x}=x.$

57. $8x-64b+27a+72\sqrt[3]{abx}=0$

58. $\frac{1+x}{1+x+\sqrt{1+x^2}}+\frac{1-x}{1-x+\sqrt{1+x^2}}=b.$

59. $\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}+\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}=b.$

60. $\sqrt[3]{x+1}+\sqrt[3]{x+2}+\sqrt[3]{x-3}=0$

61. $\sqrt[3]{x+3a}+\sqrt[3]{x+3b}+\sqrt[3]{x+3c}=0.$

62. $(a-2)\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}+(a+2)\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}=2\sqrt{a^2-4}.$

63. $\sqrt[m]{2a+x}=\sqrt[2m]{x^2-4a^2-4ax}.$

64. $\frac{x^2+9b^2}{x+3b+\sqrt{6bx}}-\frac{x^2+25b^2}{x+5b+\sqrt{10bx}}=b.$

65. $(1-b)^{\frac{1}{2}}\left(\frac{a+x}{a-x}\right)^{\frac{1}{4}}+(1+b)^{\frac{1}{2}}\left(\frac{a-x}{a+x}\right)^{\frac{1}{4}}=2(1-b^2)^{\frac{1}{4}}.$

66. $\frac{x^2+4}{x^2-2x+2}-\frac{x^4+9-3x^2}{x^2+3x+3}=24.$

67. $p\sqrt{\frac{1+x}{1-x}}+(p+2)\sqrt{\frac{1-x}{1+x}}=2\sqrt{p(p+2)}.$

68. $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}.$

69. $\sqrt[12]{x^5}-\frac{1}{2}\left(\frac{p^2-q^2}{p^2+q^2}\right)(\sqrt{x}+\sqrt[3]{x})=0.$

70. $\frac{2}{19}\left(\sqrt{x^2+39x+374}-\sqrt{x^2+20x+51}\right)=\sqrt{\frac{x+22}{x+17}}.$

142. A simple Equation cannot have more than one root. Every simple equation can be reduced to the form $ax+b=0$. If possible, let this equation have two different roots α and β.

Then we have $\left.\begin{array}{l} a\alpha+b=0 \\ a\beta+b=0 \end{array}\right\}$ ∴ by subtraction $a(\alpha-\beta)=0$.

If $a=0$, then there will be no equation and $\alpha-\beta$ is not equal to zero by supposition.

∴ *a simple equation cannot have more than one root.*

CHAPTER XXIII.

PROBLEMS LEADING TO SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

143. No general rule can be given for the solution of problems; the student must depend more or less upon his own sagacity and expertness in expressing *in algebraical language* those relations which are given in the problem. The following *HINTS*, however, will be of some use.

i. Represent the *quantity to be determined* by x and state precisely what x is chosen to represent.

ii. Express the conditions of the problem in algebraic language, using x wherever the unknown quantity enters.

iii. If one-half, one-third, one-fourth, &c., part of the unknown quantity is to be taken, then represent the unknown quantity by $2x$, $3x$, $4x$, &c, so as to avoid the introduction of fractions into the equation.

iv. Represent *odd* numbers by $2n+1$, and *even* numbers by $2n$.

v. If x represent the digit in the tens' place, its value should be denoted by $10x$; if x represent the digit in the hundreds' place, its value is denoted by $100x$.

vi. If x be the units' digit, y the tens' digit and z the hundreds' digit in a number, the number should be denoted by $100z+10y+x$.

We shall work out some examples to serve as illustrations.

Ex. 1. The sum of two numbers is 30, the greater exceeds three times the less by 2; find the numbers.

Let x = the greater number; then $30-x$ = the less.

It is given that x exceeds $3(30-x)$ by 2; hence we have $x=3(30-x)+2$. $\therefore$ $x=90-3x+2$. $\therefore$ $4x=92$.

$\therefore$ $x=23$(the *greater* number) and $30-x=7$ the *less*.

Ex. 2. Divide 46 into two parts, such that if one part be divided by 7 and the other by 3, the sum of the quotients shall be 10.

Let x = one part; then $46-x$ = the other part.

And $\frac{x}{7}+\frac{46-x}{3}$ is the sum of the quotients referred to. By the question, $\frac{x}{7}+\frac{46-x}{3}=10$. Multiply both sides by 21.

$\therefore\ 3x+7(46-x)=210.$

$\therefore\ 4x=-112.$ $\therefore\ x=28=$ *one part* and $46-x=18=$ *the other part.*

Ex. 3. A father's age is twice as great as that of his son, but 10 years ago it was three times as great, find the age of each.

Let $x=$son's age in *years*; and $2x=$father's age in years. Then by the question, $2x-10=3(x-10)$. $\therefore\ -x=-20$.

$\therefore\ x=20$ years$=$*son's* age and $2x=40=$*father's* age.

Ex. 4. There is a number of two digits whose sum is 13, and if 27 be subtracted from the number, the remainder will form a number with the digits *inverted*, find the number.

Let $x=$the digit in the units' place, then $13-x=$the digit in the tens' place.

The number$=10(13-x)+x$. The number when the digits are *reversed*$=10x+13-x$.

By the question, $10(13-x)+x-27=10x+13-x$.

$\therefore\ -10x+x-10x+x=13+27-130$

$\therefore\ -18x=-90.$ $\therefore\ x=5=$the digit in the units' place and $13-x=8=$the digit in the tens' place.

Hence 85 is the number.

Ex 5. Divide £160 among A, B and C so that A may get £10 *more* than B and B £12 *more* than C.

Let $x=$share of C in *pounds*, then $x+12=$share of B in *pounds* and $x+12+10=$share of A in *pounds*.

By the question, $x+x+12+x+12+10=160$.

$\therefore\ 3x=160-34=126.$ $\therefore\ x=42.$

$\therefore$ C's share$=$£42; B's share$=42+12=$£54; A's share $=54+10=$£64.

Ex. 6. At what time between 5 and 6 o'clock are the hands of a clock exactly *at right angles* to each other?

The minute-hand travels 12 times as fast as the hour-hand

When the hands are at right angles there must be 15 minute-divisions between them. They will be at right angles *twice* before 6 o'clock.

Let the hour-hand travel over x minute-divisions before they are at right angles *first*; then the minute-hand travels over $12x$ minute-divisions; the hour-hand then points to $12x+15$ minutes measured from XII, and also to $25+x$ minutes, for it pointed to V originally, and then travelled x minute-divisions.

Hence $12x+15=25+x$. $\therefore$ $11x=10$. $\therefore$ $x=\frac{10}{11}$.

$\therefore$ $12x=\dfrac{120}{11}=10\frac{10}{11}$ minutes past 5 o'clock.

Suppose the minute-hand has passed the hour-hand; then we see at once that $12x=25+x+15$. $\therefore$ $11x=40$. $\therefore$ $x=3\frac{7}{11}$.

$\therefore$ $12x=\dfrac{480}{11}=43\frac{7}{11}$ minutes past 5 o'clock.

Ex. 7. A workman was employed for 60 days, on condition that for every day he worked he should receive 1*s.* 3*d.*, and for every day he was idle he should forfeit 5*d.*; at the end of the time he had 20 shillings to receive; find the number of days he worked.

Let x = the number of days he worked; then

$60-x$ = ... was idle.

His pay for x days in pence $=15x$.

The sum forfeited in pence $=5(60-x)=300-5x$.

By the question, $15x-(300-5x)=20\times 12=240$. $\therefore$ $20x=540$. $\therefore$ $x=27$, the number of days he worked; $60-x=33$, the number of days he was idle.

Ex. 8. A hare is 50 leaps before a greyhound and takes 4 leaps to the greyhound's 3 leaps; but 2 of the greyhound's leaps are equal to 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let $3x$ = number of the greyhound's leaps;

then $4x$ = number of the hare's leaps in the same time;

since 2 of the greyhound's leaps = 3 of the hare's;

therefore $3x$ „ „ $=\dfrac{9x}{2}$ „ ;

By the question, $\dfrac{9x}{2}-4x=50$. $\therefore$ $\frac{1}{2}x=50$. $\therefore$ $x=100$.

$\therefore$ $3x=300$.

Ex. 9. A can do a piece of work in a days; B can do the **same** in b days; find in what time A and B will together do it.

Let x = the number of days in which A and B do the work.

Then, if W be the work, $\dfrac{W}{x}$ = work done by A and B in 1 day.

$\dfrac{W}{a}$ = work done by A in 1 day. $\dfrac{W}{b}$ = work done by B in 1 day.

$\therefore \dfrac{W}{a} + \dfrac{W}{b}$ = the work done by A and B in 1 day.

By the question, $\dfrac{W}{a} + \dfrac{W}{b} = \dfrac{W}{x}$.

$$\therefore \frac{1}{x} = \frac{1}{a} + \frac{1}{b}. \quad \therefore x = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a+b}.$$

Ex. 10. A person buys two sorts of ghee, one at Re. 1 a viss and the other at Rs. 1-8 a viss. He wishes to mix them, so that by selling the mixture at Rs. 1-3 a viss, he may gain $12\frac{1}{2}$ per cent. on the whole. What is the proportion of the mixture?

Let x = the fraction of a viss of the inferior sort in *one* viss of the mixture; then $1-x$ = the fraction of a viss of the superior sort.

The price of x viss of the inferior ghee = Rs. x.

" $1-x$ " superior " = Rs. $\frac{3}{2}(1-x)$.

$\therefore$ the price of 1 viss of the mixture $= x + \frac{3}{2}(1-x)$.

The gain $= 12\frac{1}{2}$ per cent. of the cost price $= \frac{1}{8}\{x + \frac{3}{2}(1-x)\}$.

$\therefore$ the selling price of 1 viss of the mixture $= \frac{9}{8}\{x + \frac{3}{2}(1-x)\}$.

By the question, $\frac{9}{8}\{x + \frac{3}{2}(1-x)\} = 1\frac{3}{16}$.

$\therefore 18\{x + \frac{3}{2}(1-x)\} = 19$.

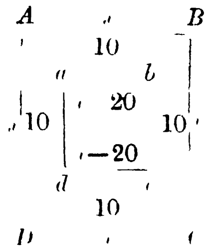
$\therefore 18x + 27 - 27x = 19. \quad \therefore -9x = -8. \quad \therefore x = \frac{8}{9}$

$\frac{8}{9}$ viss of the inferior sort and $1 - \frac{8}{9}$ or $\frac{1}{9}$ viss of the superior sort.

$\therefore$ superior : inferior $= \frac{1}{9} : \frac{8}{9} = 1 : 8$.

Ex. 11. An officer can form his men into a *hollow* square 10 deep. The number of men in the regiment is 2,800. Find the number of men in the front of the hollow square.

Let x = the number of men in the front of the hollow; then the number of men in the square $= x^2 - (x-20)^2$.



the question,

$x^2 - (x-20)^2 = 2800$

$\therefore\ 40x - 400 = 2800$

$\therefore\ 40x = 3200$

$\therefore\ x = 80.$

Note.—*ABCD* is a solid square having x men in the front; *abcd* is another having $x-20$ men in the front; and *ABCD*−*abcd* is a *hollow* square having x men in the front but only 10 deep.

Ex. 12. A man rides one-third of the distance from *A* to *B* at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour. If he had travelled at a uniform rate of $3c$ miles per hour he could have ridden from *A* to *B* and back again in the same time. Prove that $\frac{1}{a}+\frac{1}{b}=\frac{2}{c}$.

Let x = the distance from *A* to *B* in miles,

then $\frac{\frac{1}{3}x}{a}$ = time in hours taken to ride $\frac{1}{3}$ of the distance.

$\frac{\frac{2}{3}x}{2b}$ = " " " $\frac{2}{3}$ " "

and $\frac{2x}{3c}$ = " " " from *A* to *B* and back again. By the question, $\frac{\frac{1}{3}x}{a}+\frac{\frac{2}{3}x}{2b}=\frac{2x}{3c}$.

$\therefore\ \frac{x}{3a}+\frac{2x}{6b}=\frac{2x}{3c}$ $\quad\therefore\ \frac{1}{a}+\frac{1}{b}=\frac{2}{c}$.

EXERCISE 60.

1. Find a number from which if 6 be taken and the remainder multiplied by 6, and then 6 added to the product, this sum divided by 3 shall give the required number.

2. Find two numbers the difference of which is 14, and the greater number being divided by the less is 8.

3. How many trees are there in a garden containing one-fifth mango-trees, three-sevenths apple-trees and 26 trees of various other kinds?

4. A house and garden cost Rs. 850 and the price of the garden is equal to $\frac{5}{12}$ths the price of the house. Find the price of each.

5. A can do a piece of work in 50 days, B in 60 days and C in 75 days. In what time will they do it all working together?

6. Two persons A and B own a flock of sheep. They agree to divide its value. A takes 72 sheep, while B takes 92 sheep and pays A £35. Required the value of a sheep.

7. A can do a piece of work in 4 hours, C in $3\frac{3}{5}$ hours and B in $5\frac{1}{7}$ hours; in what time will they do the work together?

8. Divide 19 guineas among 3 persons, so that the first shall have twice as much as the second, and the third 5*s.* less than the second.

9. At what time between 7 and 8 o'clock will the hour and minute hands of a clock be (1) together, (2) at right angles, (3) in a straight line.

10. The ages of two men differ by 10 years, and 15 years ago the elder was just twice as old as the younger. Find the ages of the men.

11. A man has a labourer on this condition, that for every day he worked he should receive 2*s.*; but that for every day he was absent he was to forfeit 1*s.* 4*d.*; when 390 days were past, neither of them was indebted to the other. How many days did the man work?

12. A person bought a certain number of eggs at 2 a penny and as many at 3 a penny, and sold them at the rate of 5 for 2*d.* losing 4*d.* by the bargain. How many eggs did he buy?

13. How much tea at 4*s.* 6*d.* per lb. must be mixed with 50 lbs. at 6*s.* per lb. so that the mixture may be worth 5*s.* 6*d.* per lb.?

14. From two bags containing the same number of coins, sums are taken in the ratio of 4 : 5. And if 12 more coins

were taken from the bag that now has the fewer, the number of coins taken from it would be double that taken from the other. How many coins were taken from each?

15. A certain fraction is equal to $\frac{2}{3}$; when its numerator is increased by 5 and its denominator by 9 it becomes $\frac{5}{8}$. Find the fraction.

16. A person bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each. If he had expended his money equally on the two kinds, he would have had 2 sheep more than he did. How many did he buy?

17. A labourer is engaged for n days, on condition that he receives p pence for every day he works, and pays q pence for every day he is idle. At the end of the time he receives a pence. How many days did he work and how many was he idle?

18. A person has just 2 hours at his disposal. How far may he ride in a coach which travels 12 miles an hour, so as to return home in time, walking back at the rate of 4 miles an hour?

19. There is a number composed of two figures, of which the figure in the units' place is 3 times that in the tens', and if 36 be added to the number, the sum is expressed by the same digits *reversed*. What is the number?

20. The length of a field is twice its breadth; and another field which is 50 yards longer and 10 yards broader, contains 6,800 square yards more than the former. Find the dimensions of each.

21. The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet and the breadth diminished by 2 feet, the area would not have been altered. Find the dimensions.

22. A number consists of two digits; the sum of the digits is 8, and if the left digit be diminished by 2 it will be equal to $\frac{3}{7}$ of the number. Find the number.

23. A number consists of two digits whose sum is 5; if 10 times the digit in the tens' place be added to 4 times the digit in the units' place, the number will be *inverted*. What is the number?

24. If a train, which travels at the rate of 35 miles an hour, start one-quarter of an hour after a luggage train, and overtake it in 10 minutes, find the speed of the luggage train.

25. Two passengers are charged for excess of luggage 2*s*. 10*d*. and 7*s*. 6*d*. respectively; had the luggage all belonged to one of them he would have been charged for excess 14*s*. 6*d*. How much would they have been charged if none had been allowed free?

26. Two persons A and B were engaged in counting a number of sovereigns, and A counted three for every two counted by B; when B had counted 22 he forgot his reckoning and was obliged to recommence, and when he had counted 64 there were no more left to count. Find the number of sovereigns.

27. Two persons walk at the rate of 5 and 6 miles an hour respectively; they set out to meet each other from two places 22 miles apart. Having passed each other once, find the place of their second meeting, supposing them to continue their journey between the two places. Also find the time when the second meeting takes place.

28. Divide the number A into 4 parts, such that if to the first you add B, from the second subtract B, multiply the third by B, and divide the fourth by B, the results will be all equal. If $A=90$, $B=2$, what will the results be?

29. A greyhound spying a hare at the distance of 60 of his own leaps from him pursues her, making 4 leaps for every 5 leaps of the hare, but passing over as much ground in 3 leaps as the hare does in 4. How many leaps did each make during the whole course?

30. At the review of an army the troops were drawn up into a solid mass 40 deep, when there were just $\frac{1}{4}$th as many men in front as there were spectators. Had the depth however been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Find the number of men in the army.

31. Find a number of three digits, each greater by unity than that which follows it, so that its excess above $\frac{1}{4}$th of the number formed by inverting the digits shall be 36 times the sum of the digits.

32. A, B, C travel from the same place at the rate of 4, 5, 6 miles an hour respectively and B starts 2 hours after A. How long after B must C start in order that they may both overtake A at the same instant?

33. A train which travels at the uniform rate of 40 miles an hour meets a person walking along the line in the opposite

direction at the rate of 4 miles an hour, and passes him in $5\frac{5}{6}$ seconds. Find the length of the train.

34. Two trains running at the rates of 25 and 20 miles respectively on parallel rails in opposite directions are observed to pass each other in 8 seconds; and when they are running in the same direction at the same rates as before, a person sitting in the faster train observes that he passes the other in $31\frac{1}{2}$ seconds. Find the length of the trains.

35. A garrison of 1,500 men have provisions for 36 days; but after 16 days it was reinforced; and the provisions were exhausted in 12 days. Find the number of men in the reinforcement.

36. An officer on attempting to draw up his regiment in the form of a solid square finds that he has 31 men over and that he would require 24 men more in order to increase the side of the square by 1 man. How many men were there in the regiment.

37. At a contested election, 1,793 votes are polled, and the defeated candidate is left in a minority of 313. Find the number of votes for each candidate.

38. Two coaches start at the same time from York and London, a distance of 200 miles; the one from London travels at $9\frac{1}{2}$ miles an hour, and that from York at $10\frac{1}{2}$ miles. Where and when will they meet?

39. Two men set out on a journey, walking at the rate of 4 miles an hour; after walking for six hours, one of the two lessens his rate to 3 miles an hour; the other continues on at the same rate, and arrives at the end of his journey an hour before his companion. Find the length of their journey.

40. A rower who can pull at the rate of 6 miles an hour can pull 10 miles down the river in half the time that he will take to pull 10 miles up it. Find the rate at which the river flows.

41. A railroad runs from A to C; a goods' train starts from A at 12 o'clock and a passenger train at 1 o'clock; after going two-thirds of the distance the goods' train breaks down, and can only travel at three-fourths of its former speed. At 20 minutes before 3, a collision occurs, 10 miles from C. The rate of the passenger train being double of the diminished speed of the goods' train, find the distance from A to C, and the rates of the two trains in miles per hour.

42. A luggage train going at the rate of 10 miles an hour is some distance in advance of an express engine, which starts

to overtake it, and just comes up with it in 5 hours; had the express engine travelled 10 miles an hour less, it would have been $7\frac{1}{2}$ hours in coming up with the luggage train. Find the distance between the train and the express engine at first, and the rate at which the latter travelled.

43. A landlord let his farm for £10 a year in money, and a corn-rent. When corn sold at 10*s.* a bushel he received at the rate of 10*s.* an acre for his land, but when it sold at 13*s.* 6*d.* a bushel, 13*s.* an acre. Of how many bushels did the corn-rent consist?

44. A person buys some tea at 3*s.* a lb. and some at 5*s.* a lb.; he wishes to mix them so that by selling the mixture at 3*s.* 8*d.* a lb. he may gain 10 per cent. on his outlay. In what proportion should he mix them?

45. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former. Find the number of men.

46. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head; after being at sea 20 days, she encounters a storm in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ths of a lb. Find the original number of the crew.

47. A person rows from Cambridge to Ely—, a distance of 20 miles—, and back again in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

48. On a certain morning mangoes were sold at a certain price per score: the next morning as many mangoes could be bought for one rupee as scores for Rs. 30 the day before the whole price of 30 mangoes, 15 bought one day and 15 the other, was 12 as. 6 p. Find the price of a mango on each day

49. To complete a certain work A requires m times as many days as B and C together; B requires n times as many days as A and C together and C requires p times as many days as A and B together. Prove that $\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1$.

CHAPTER XXIV.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

144. Suppose we have an equation of the form $ax+by=c$ where x and y are both unknown quantities, and a, b, c known quantities, it is clear that we shall obtain from it, $x=\dfrac{c-by}{a}$; and since y is an unknown quantity, the value of x must still be unknown also. But if we have a second equation $px+qy=r$ where x and y have the same values as in the first equation, we may reduce this second equation to the form $x=\dfrac{r-qy}{p}$ (p, q, r are known quantities).

Since the value of x is the same in both equations, we have $\dfrac{c-by}{a}=\dfrac{r-qy}{p}$; an equation containing only one unknown quantity y, from which therefore the value of y may be found in terms of a, b, c, p, q, r which are known quantities, and x can be determined by substituting the value of y thus found in either of the equations $ax+by=c$ and $px+qy=r$.

As these equations are satisfied by the *same* values of the unknown quantities, they are called **Simultaneous Equations**.

Hence it follows that if we have two unknown quantities, we must have two independent equations.

145. There are *three* methods by which such equations can be solved.

Let the equations be $4x+3y=31$ and $3x+2y=22$.

First Method. Equalize the co-efficients of *one* of the unknown quantities.

$$4x+3y=31\ldots\ldots(1) \qquad 3x+2y=22\ldots\ldots(2).$$

The L.C.M. of the co-efficients of y is 6, and of those of x is 12, so that it is easier to equalize the co-efficients of y than those of x.

Multiplying (1) by 2, we have $8x+6y=62\ldots\ldots$ (3).

Multiplying (2) by 3, we have $9x+6y=66\ldots\ldots$ (4).

Subtracting (3) from (4), we have $x=4$.

Substituting 4 for x in (1), we get $4\times 4+3y=31$.

$\therefore 3y=15. \qquad \therefore y=5.$

The required solution is $x=4$ and $y=5$.

NOTE.– **This method is the one generally employed.**

Second Method. Find x in terms of y, or y in terms of x from both equations, and equate the values so obtained.

$$4x+3y=31\ldots\ldots(1). \qquad 3x+2y=22\ldots\ldots(2).$$

From (1), $4x=31-3y$. $\quad \therefore x=\dfrac{31-3y}{4}\ldots\ldots(3)$.

From (2), $3x=22-2y$. $\quad \therefore x=\dfrac{22-2y}{3}\ldots\ldots(4)$.

From (3) and (4), $\dfrac{31-3y}{4}=\dfrac{22-2y}{3}$.

$\therefore 93-9y=88-8y. \qquad \therefore -y=-5. \qquad \therefore y=5.$

From (3), $x=\dfrac{31-15}{4}=4$.

Third Method. Find x in terms of y, or y in terms of x from one of the equations, and substitute the value so obtained in the other.

$$4x+3y=31\ldots\ldots\ldots\ldots\ldots(1). \qquad 3x+2y=22\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), $3y=31-4x$. $\quad \therefore y=\dfrac{31-4x}{3}\ldots\ldots\ldots\ldots\ldots(3)$.

Substituting this value of y in (2), we get $3x+\dfrac{2(31-4x)}{3}=22$. $\quad \therefore 9x+62-8x=66 \quad \therefore x=4$.

From (3), $y=\dfrac{31-16}{3}=5$.

EXAMPLES WORKED OUT.

1. Solve the equations $5x-8y=12\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$.
$$11x+12y=56\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying (1) by 3, we have $15x-24y=36\ldots\ldots\ldots(3)$.

Multiplying (2) by 2, we have $22x+24y=112\ldots\ldots\ldots(4)$.

Adding (3) and (4), $37x=148$. $\quad \therefore x=4$.

Substituting 4 for x in (1), we get—

$$20-8y=12. \qquad \therefore -8y=-8. \qquad \therefore y=1.$$

2. Solve the equations $ax + by = c$..................(1).

$a'x + b'y = c'$..............(2).

Multiplying (1) by b', $ab'x + bb'y = cb'$.........(3).

Multiplying (2) by b, $a'bx + bb'y = c'b$.........(4).

Subtracting (4) from (3), $x(ab' - a'b) = cb' - c'b$.

$$\therefore\ x = \frac{cb' - c'b}{ab' - a'b}$$

Instead of substituting this value of x in either of the equations, it is easier to get the value of y by the same method.

Thus from (1), $a'ax + a'by = a'c$

from (2), $a'ax + ab'y = ac'$.

Subtracting, $y(a'b - ab') = a'c - ac'$. $\therefore\ y = \dfrac{a'c - ac'}{a'b - ab'}$.

3. Solve the equations—

$$\frac{x+y}{3} + \frac{x-y}{4} = 11,\quad \frac{x+y}{2} - \frac{x-y}{3} = 8.$$

Clearing of fractions, we get—

$4x + 4y + 3x - 3y = 132$, or $7x + y = 132$...(1).

and $3x + 3y - 2x + 2y = 48$, or, $x + 5y = 48$...(2).

From (1), $35x + 5y = 660$......(3) From (2), $x + 5y = 48$...(4).

Subtracting, $34x = 612$. $\therefore\ x = 18$.

Substituting 18 for x in (2), we get—

$18 + 5y = 48$. $\therefore\ 5y = 30$. $\therefore\ y = 6$.

4. Solve the equations—

$2x + \cdot 4y = 1\cdot 2$...(1). $3\cdot 4x - \cdot 02y = \cdot 01$..........(2).

From (1) $\times 3\cdot 4$, we have $6\cdot 8x + 1\cdot 36y = 4\cdot 08$ ...(3).

From (2) $\times 2$, we have $6\cdot 8x - \cdot 04y = \cdot 02$ ...(4).

Subtracting, $1\cdot 4y = 4\cdot 06$. $\therefore\ y = \dfrac{4\cdot 06}{1\cdot 4} = 2\cdot 9$.

Substituting, $2\cdot 9$ for y in (1), we get $2x + 1\cdot 16 = 1\cdot 2$.

$\therefore\ 2x = \cdot 04$. $\therefore\ x = \cdot 02$.

5. Solve the equations—

$$\frac{a}{x} + \frac{b}{y} = m\text{.........(1)}.\qquad \frac{b}{x} + \frac{a}{y} = n\text{.........(2)}.$$

Here the unknown quantities are $\frac{1}{x}$ and $\frac{1}{y}$.

From (1) $\times b$, $\frac{ab}{x}+\frac{b^2}{y}=bm$ ... ... (3).

From (2) $\times a$, $\frac{ab}{x}+\frac{a^2}{y}=an$ ... ... (4).

Subtracting, $\frac{1}{y}(b^2-a^2)=bm-an$.

$$\therefore \frac{1}{y}=\frac{bm-an}{b^2-a^2}. \qquad \therefore y=\frac{b^2-a^2}{bm-an}.$$

Similarly the value of $\frac{1}{x}$ can be got; hence the value of x.

6. Solve the equations $(a+c)x-(b+d)y=c$ ... ...(1)
$(a+d)x-(b+c)y=d$ ... ...(2)

Subtracting, $x(c-d)+y(c-d)=c-d$. $\therefore x+y=1$...(3).

$$\therefore (b+d)x+(b+d)y=b+d.$$

and $(a+c)x-(b+d)y=c$

Adding $x(a+b+c+d)=b+d+c$

$$\therefore x=\frac{b+c+d}{a+b+c+d}.$$

From (3), $y=1-x=1-\frac{b+c+d}{a+b+c+d}=\frac{a}{a+b+c+d}$.

EXERCISE 61.

Solve the following equations: –

1. $\left.\begin{array}{l} x+y=8 \\ x-y=2 \end{array}\right\}$.

2. $\left.\begin{array}{l} 4x+3y=19 \\ 11x-2y=1 \end{array}\right\}$.

3. $\left.\begin{array}{l} \frac{x+y}{3}+2x=13 \\ \frac{x-y}{2}+y=\text{[illegible]} \end{array}\right\}$.

4. $\left.\begin{array}{l} 17x-10y=31 \\ 17x+9y=69 \end{array}\right\}$.

5. $\left.\begin{array}{l} \frac{3x}{5}+\frac{y}{4}=13 \\ \frac{x}{3}-\frac{y}{8}=3 \end{array}\right\}$.

6. $\left.\begin{array}{l} \frac{x+y}{3}-\frac{x-y}{5}=0 \\ \frac{x+y}{7}-\frac{x-y}{35}+\frac{1}{7}=0 \end{array}\right\}$.

7. $\left.\begin{aligned} 2x - \frac{y-3}{5} &= 4 \\ 3y = 9 - \frac{x-2}{3} \end{aligned}\right\}$

8. $\left.\begin{aligned} x(y+7) &= y(x+1) \\ 2x &= 3y - 19 \end{aligned}\right\}.$

9. $\left.\begin{aligned} ax + by &= c \\ px + qy &= r \end{aligned}\right\}.$

10. $\left.\begin{aligned} ax + by &= c \\ bx - ay &= d \end{aligned}\right\}.$

11. $\left.\begin{aligned} ax - by &= 0 \\ x + y &= c \end{aligned}\right\}.$

12. $\left.\begin{aligned} ax - by &= m \\ cx + dy &= n \end{aligned}\right\}.$

13. $\left.\begin{aligned} \frac{3}{y} - \frac{1}{x} &= 1 \\ \frac{2}{5x} + \frac{5}{2y} &= 7 \end{aligned}\right\}$

14. $\left.\begin{aligned} \frac{x}{a} - \frac{y}{b} &= m \\ \frac{x}{c} + \frac{y}{d} &= n \end{aligned}\right\}.$

15. $\cdot 3x + \cdot 125y = 3x - y$; $3x - \cdot 5y = 2\cdot 25 - 3y$.

16. $\left.\begin{aligned} 4x + 8y &= 2\cdot 4 \\ 10\cdot 2x - \cdot 06y &= \cdot 03 \end{aligned}\right\}$

17. $\left.\begin{aligned} 7\cdot 2x + 3\cdot 6y &= 54 \\ 2\cdot 3x + 6\cdot 9y &= 23 \end{aligned}\right\}.$

18. $\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 - \frac{x}{c} \\ \frac{y}{a} + \frac{x}{b} &= 1 + \frac{y}{c} \end{aligned}\right\}$

19. $\left.\begin{aligned} x - \frac{2y-x}{23-x} &= 20 - \frac{59-2x}{2} \\ y + \frac{y-3}{x-18} &= 30 - \frac{73-3y}{3} \end{aligned}\right\}.$

20. $\left.\begin{aligned} 2x - \frac{y-3}{5} &= \frac{5x-2}{2} \\ 2y - \frac{x-5}{3} &= \frac{7y-7}{2} \end{aligned}\right\}$

21. $\left.\begin{aligned} (x+5)(y+7) &= (x+1) \\ &\quad \times (y-9) + 112 \\ 2x + 10 &= 3y + 1 \end{aligned}\right\}.$

22. $\left.\begin{aligned} a(x+y) + b(x-y) &= 1 \\ c(x+y) + d(x-y) &= 1 \end{aligned}\right\}$

23. $\left.\begin{aligned} axy &= c(bx + ay) \\ bxy &= c(ax - by) \end{aligned}\right\}.$

24. $\left.\begin{aligned} a^2x + bcy &= b^2 \\ ax + by &= c \end{aligned}\right\}.$

25. $\left.\begin{aligned} a^2x + b^2y &= n \\ b^2x + a^2y &= n \end{aligned}\right\}.$

26. $\left.\begin{aligned} \frac{a+x}{ax} - \frac{b+y}{by} &= \frac{a+b}{ab} \\ \frac{1}{x} + \frac{1}{y} &= \frac{2}{a} \end{aligned}\right\}.$

27. $\left.\begin{aligned} \frac{a^2}{x} + \frac{b^2}{y} &= \frac{(a+b)^2}{x+y} \\ x + y &= \frac{4b}{a} \end{aligned}\right\}.$

28. $\left.\begin{aligned} (a+b)x + (a-b)y &= 2a \\ (a-b)x + (a+b)y &= 2a \end{aligned}\right\}$

29. $\left.\begin{aligned} \frac{x}{a+b} + \frac{y}{a-b} &= 2 \\ x + y &= 2a \end{aligned}\right\}.$

30. $\left.\begin{array}{l}\dfrac{2a}{x}-\dfrac{b}{y}=1\\[2mm]\dfrac{2b}{y}-\dfrac{a}{x}=1\end{array}\right\}$

31. $\left.\begin{array}{l}\dfrac{a+b}{x}+\dfrac{a-b}{y}=2a\\[2mm]\dfrac{3}{x}+\dfrac{4}{y}=7\end{array}\right\}.$

32. $\left.\begin{array}{l}\dfrac{2}{x}+\dfrac{3}{y}=1\\[2mm]\dfrac{7}{x}+\dfrac{4}{y}=\dfrac{15}{8}\end{array}\right\}.$

33. $\left.\begin{array}{l}\dfrac{x}{4}+\dfrac{2}{y}=2\\[2mm]\dfrac{2x}{5}+\dfrac{3}{2y}=\dfrac{47}{20}\end{array}\right\}.$

34. $\left.\begin{array}{l}ax+by=c^2\\[2mm]\dfrac{a}{b+y}-\dfrac{b}{a+x}=0\end{array}\right\}.$

35. $\left.\begin{array}{l}9x+8y=43xy\\8x+9y=42xy\end{array}\right\}.$

36. $(a-d)x+(b-d)y=(b-a)(c-d)\,;\ ax+by=(b-a)c.$

37. $x+y+1=0\,;\ a^2x+b^2y+c(a+b)=ab.$

38. $(b-a)x-(c-b)y=c-a\,;\ (a-c)x-(b-a)y=b-c.$

39. $\left.\begin{array}{l}(b+c-a)x+(c+a-b)y=2c\\(c+a-b)x+(a+b-c)y=2a\end{array}\right\}.$

40. $ax+by=a^3+b^3\,;\ (b-c)x+(c-a)y=(b-c)a^2+(c-a)b^2.$

41. $\left.\begin{array}{l}(a+b)x-(a-b)y=\dfrac{a+5b}{5}(x^2-y^2)\\[2mm](a-b)x+(a+b)y=\dfrac{5a-b}{5}(x^2-y^2)\end{array}\right\}.$

42. $\sqrt{y}-\sqrt{a-x}=\sqrt{y-x}\,;\ 2\sqrt{y-x}+9\sqrt{a-x}=5\sqrt{a+x}.$

43. $\sqrt{x}-\sqrt{a+y}=\sqrt{x+y}\,;\ 3\sqrt{a+x}+2\sqrt{x+y}=9\sqrt{a+x}.$

44. $\dfrac{a+b}{x}+\dfrac{a-b}{y}=\dfrac{2a}{a^2-b^2}\,;\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{2(a^2+b^2)}{(a^2-b^2)^2}.$

45. $\left.\begin{array}{l}(6a+b)x+(a+6b)y=\dfrac{xy(a+b)}{49}\\[2mm](9a-2b)x-(2a-9b)y=\dfrac{xy(a+b)}{49}\end{array}\right\}.$

46. $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2\,;\ b\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=a\left(\dfrac{1}{y}-\dfrac{1}{x}\right).$

47. $x+y=a+b\,;\quad (a+2b)^3x-(2a+b)^3y=(a-b)^3(a+b).$

48. $\left.\begin{aligned}\frac{x}{a^2+1}+\frac{y}{b^2+1}&=\frac{1}{a^2-b^2}\\ \frac{x}{a^2-1}-\frac{y}{b^2-1}&=\frac{1}{a^2+b^2}\end{aligned}\right\}.$ 49. $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=1\\ \frac{b}{x}+\frac{a}{y}&=\frac{a^3}{bxy}\end{aligned}\right\}.$

50. $\left.\begin{aligned}(pa+qb)x+(pa^2+qb^2)y&=pa^3+qb^3\\ (pa^2+qb^2)x+(pa^3+qb^3)y&=pa^4+qb^4\end{aligned}\right\}.$

146. Simultaneous equations of the first degree involving three or more unknown quantities. If we have three independent equations, involving three unknown quantities, for example,

$$ax+by+cz=m\text{.................................}(1),$$
$$a_1x+b_1y+c_1z=n\text{.................................}(2),$$
$$a_2x+b_2y+c_2z=p\text{.................................}(3),$$

the values of x, y and z may be determined thus:—

Multiplying (1) by a_1, $aa_1x+a_1by+a_1cz=a_1m$.
Multiplying (2) by a, $aa_1x+ab_1y+ac_1z=an$.

$$\therefore\ y(a_1b-ab_1)+z(a_1c-ac_1)=a_1m-an\text{............}(4).$$

Again, multiplying (1) by a_2, $aa_2x+a_2by+a_2cz=a_2m$
Multiplying (3) by a, $aa_2x+ab_2y+ac_2z=ap$.

$$\therefore\ y(a_2b-ab_2)+z(a_2c-ac_2)=a_2m-ap\text{............}(5).$$

The values of y and z may be determined from the equations (4) and (5); and by substituting their values so obtained in any of the given equations the value of x can be determined.

NOTE.—*This method may be extended to equations containing four or more unknown quantities.*

Ex. 1. Solve the equations—

$$4x-2y+5z=18\text{..............}(1).$$
$$2x+4y-3z=22\text{..............}(2).$$
$$6x+7y-z=63\text{..............}(3).$$

From (1), $4x-2y+5z=18$.
From (2) × 2, $4x+8y-6z=44$.
Subtracting, $10y-11z=26$............(4).
From (2) × 3, $6x+12y-9z=66$.

From (3), $6x+7y-z=63$.
Subtracting, $5y-8z=3$............(5).
From (4), $10y-11z=26$.
From (5)×2, $10y-16z=6$.
$\therefore\ 5z=20$. $\therefore\ z=4$.
From (4), $10y=26+11z=26+44=70$. $\therefore\ y=7$.
From (1), $4x=18+2y-5z=18+14-20=12$. $\therefore x=3$.

Ex. 2. Solve the equations—

$x+y=c$......(1). $y+z=a$......(2). $z+x=b$......(3).

Adding (1), (2) and (3), $2(x+y+z)=a+b+c$.

$\therefore\ x+y+z=\dfrac{a+b+c}{2}$...........(4).

Subtracting (1) from (4), $z=\dfrac{a+b+c}{2}-c=\dfrac{a+b-c}{2}$

...............(2)..........., $x=\dfrac{a+b+c}{2}-a=\dfrac{b+c-a}{2}$

...............(3)...........,$y=\dfrac{a+b+c}{2}-b=\dfrac{a+c-b}{2}$.

EXERCISE 62.

Solve the following equations :—

1. $x+y+z=6$
$x+2y+3z=14$
$x+3y+4z=19$.

2. $x+y-z=0$
$2x+3y-4z=-4$
$3x-2y-3z=-10$.

3. $x+y+z=90$
$2x-3y=-20$
$2x-4z=-30$.

4. $2x+3y+4z=29$
$3x+2y+5z=32$
$4x+3y+2z=25$.

5. $5x+3y=65$
$2y-z=11$
$3x+4z=57$.

6. $\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}=62$
$\dfrac{x}{3}+\dfrac{y}{4}+\dfrac{z}{5}=47$
$\dfrac{x}{4}+\dfrac{y}{5}+\dfrac{z}{6}=38$.

7. $x+y=3$
$y+z=5$
$z+x=4$.

8. $x+y-z=6$
$z+x-y=10$
$y+z-x=14$.

9. $y+\frac{z}{3}=\frac{x}{5}+5$
$\frac{x-1}{4}-\frac{y-2}{5}=\frac{z+3}{10}$
$x-\frac{2y-5}{3}=1\frac{3}{4}-\frac{z}{12}.$

10. $\frac{x}{2}-\frac{y}{3}+\frac{z}{4}=22$
$\frac{x}{3}+\frac{y}{4}-\frac{z}{5}=-1$
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.$

11. $\frac{1}{x}-\frac{5}{y}+\frac{1}{z}=-3.$
$\frac{3}{x}+\frac{4}{y}-\frac{2}{z}=5$
$\frac{5}{x}+\frac{7}{y}+\frac{6}{z}=18.$

12. $\frac{a}{x}+\frac{b}{y}=\frac{1}{m}$
$\frac{b}{y}+\frac{c}{z}=\frac{1}{n}$
$\frac{c}{z}+\frac{a}{x}=\frac{1}{p}.$

13. $\frac{1}{x}+\frac{1}{y}$ or $\frac{x+y}{xy}=70$
$\frac{1}{y}+\frac{1}{z}$ or $\frac{y+z}{yz}=140$
$\frac{1}{z}+\frac{1}{x}$ or $\frac{z+x}{zx}=84.$

14. $\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=\frac{38}{5}$
$\frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=\frac{61}{6}$
$\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=\frac{161}{10}.$

15. $\frac{2}{x}-\frac{1}{y}=\frac{3}{z}$
$\frac{3}{z}+\frac{2}{y}=2$
$\frac{1}{x}+\frac{1}{z}=\frac{4}{3}.$

16. $x+\frac{1}{2}(y+z)=y+\frac{2}{3}(x+z)$
$=z+\frac{3}{4}(x+y)=x+y+z-4.$

17. $\frac{6y-4x}{3z-7}=\frac{5z-x}{2y-3z}=\frac{y-4z}{3y-2x}=1.$

18. $x+y+z=0$
$(a+b)x+(a+c)y+(b+c)z=0$
$abx+acy+bcz=1.$

19. $xy=a(x+y)$
$yz=c(y+z)$
$zx=b(x+z).$

20. $x-ay+a^2z=a^3$
$x-by+b^2z=b^3$
$x-cy+c^2z=c^3.$

21. $bx+ay=c$
$cx+az=b$
$cy+bz=a.$

22. $\frac{a}{y}+\frac{b}{z}=\frac{a}{z}+\frac{b}{x}=\frac{a}{x}+\frac{b}{y}=c.$

23. $3(x-z)=by$; $2(x-y)=z$; $(y-z)=1$.

24. $xyz=a(yz-xz-xy)=b(xz-yz-xy)=c(xy-yz-xz)$.

25. $ax+by+cz=a+b+c$; $x+y+z=3$; $ay+bx=a+b$.

26. $\dfrac{ay+bx}{c}=\dfrac{bz+cy}{a}=\dfrac{az+cx}{b}=1$.

27. $x+y+z+u=10$; $x-2y+3z=6$; $y+z-2u=-3$; and $3x-2z+4u=13$.

28. $x+y+z+u=0$; $15u+13x=-(11y+9z)$, $71u+31y=-(47x+23z)$; $1+105u=-(45x+21y)$.

29. $x+y+z+u=10$; $x+y+z-u=2$; $y+z+u-x=8$; $z+x+u-y=6$.

30. $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=6$; $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{u}=7$, $\dfrac{1}{x}+\dfrac{1}{z}+\dfrac{1}{u}=8$,

$$\frac{1}{y}+\frac{1}{z}+\frac{1}{u}=9.$$

147. Method of Cross Multiplication.

If $a_1x+b_1y+c_1z=0$ and $a_2x+b_2y+c_2z=0$, shew that

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Multiply (1) by a_2 ; $a_1a_2x+a_2b_1y+a_2c_1z=0$.

Multiply (2) by a_1 ; $a_1a_2x+a_1b_2y+a_1c_2z=0$.

Subtracting, $y(a_2b_1-a_1b_2)+z(a_2c_1-a_1c_2)=0$.

$\therefore\ y(a_2b_1-a_1b_2)=-z(a_2c_1-a_1c_2)$.

$$\therefore\ \frac{y}{c_1a_2-a_1c_2}=\frac{z}{a_1b_2-a_2b_1}\ \ldots\ldots(1).$$

Again, multiply (1) by b_2 ; $a_1b_2x+b_1b_2y+b_2c_1z=0$.

Multiply (2) by b_1 ; $a_2b_1x+b_1b_2y+b_1c_2z=0$.

Subtracting, $x(a_1b_2-a_2b_1)+z(b_2c_1-b_1c_2)=0$.

$\therefore\ x(a_1b_2-a_2b_1)=-z(b_2c_1-b_1c_2)$.

$$\therefore\ \frac{x}{b_1c_2-b_2c_1}=\frac{z}{a_1b_2-a_2b_1}\ \ldots\ldots\ldots(2).$$

Hence from (1) and (2),

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

These results can be easily remembered by writing down the co-efficients thus :—

$$\begin{array}{l} a_1 b_1 c_1 a_1 . \\ a_2 b_2 c_2 a_2 . \end{array}$$

The quantity under x is obtained by writing the co-efficients of y and z and multiplying across thus :—

$$\begin{array}{cc} b_1 & c_1 \\ b_2 - & + c_2 \end{array} = b_1 c_2 - c_1 b_2 .$$

The quantity under y is obtained in the same way from the co-efficients of z and x.

$$\begin{array}{cc} c_1 & a_1 \\ c_2 - & + a_2 \end{array} = c_1 a_2 - a_1 c_2$$

The quantity under z is obtained in the same way from the co-efficients of x and y.

$$\begin{array}{cc} a_1 & b_1 \\ a_2 - & + b_2 \end{array} = a_1 b_2 - b_1 a_2 .$$

Note.—If $z=1$ in the above equations we have $\dfrac{x}{b_1 c_2 - b_2 c_1} = \dfrac{y}{c_1 a_2 - c_2 a_1} = \dfrac{1}{a_1 b_2 - a_2 b_1}$, which gives the solution of the equations.

$$a_1 x + b_1 y + c_1 = 0 \text{ and } a_2 x + b_2 y + c_2 = 0.$$

If the above values are committed to memory, it will enable us to solve readily simple equations involving two unknown quantities.

Ex. 1. Solve $5y-3x=9$; $5x+2y=16$.

Writing the equations thus: $3x-5y+9=0$. . . (1).

$5x+2y-16=0$. . . (2).

Here $a_1=3$, $b_1=-5$, $c_1=9$ and $a_2=5$, $b_2=2$, $c_2=-16$.

$$\therefore \frac{x}{(-5)(-16)-9\times 2}=\frac{y}{9\times 5-3(-16)}=\frac{1}{3\times 2-(-5)(5)},$$

or $\dfrac{x}{80-18}=\dfrac{y}{45+48}=\dfrac{1}{6+25}$. $\quad\therefore \dfrac{x}{62}=\dfrac{y}{93}=\dfrac{1}{31}$.

$$\therefore x=\frac{62}{31}=2 \text{ and } y=\frac{93}{31}=3.$$

Ex. 2. Solve $\dfrac{3}{x}+\dfrac{4}{y}=2$; $\dfrac{2}{x}+\dfrac{3}{y}=\dfrac{17}{12}$.

Write the equations thus: $3\left(\dfrac{1}{x}\right)+4\left(\dfrac{1}{y}\right)-2=0$......(1).

$2\left(\dfrac{1}{x}\right)+3\left(\dfrac{1}{y}\right)-\dfrac{17}{12}=0$......(2).

$$\text{Hence } \frac{\frac{1}{x}}{4\left(-\frac{17}{12}\right)-(-2)(3)}=\frac{\frac{1}{y}}{(-2)(2)-3\left(-\frac{17}{12}\right)}$$

$$=\frac{1}{3\times 3-2\times 4}, \text{ or } \frac{\frac{1}{x}}{-\frac{17}{3}+6}=\frac{\frac{1}{y}}{-4+\frac{17}{4}}=\frac{1}{9-8},$$

or $\dfrac{\frac{1}{x}}{\frac{1}{3}}=\dfrac{\frac{1}{y}}{\frac{1}{4}}=1$, or $\dfrac{1}{x}=\dfrac{1}{3}$ or $x=3$, and $\dfrac{1}{y}=\dfrac{1}{4}$ or $y=4$.

Ex. 3. Solve $x-2y+z=0$.........(1).

$9x-8y+3z=0$.........(2).

$2x+3y+5z=36$.........(3).

From (1) and (2), $\dfrac{x}{(-2)(3)-(1)(-8)}=\dfrac{y}{1\times9-1\times3}$

$=\dfrac{z}{1(-8)-(-2)(9)}$, or $\dfrac{x}{-6+8}=\dfrac{y}{9-3}=\dfrac{z}{-8+18}$, or $\dfrac{x}{2}$

$=\dfrac{y}{6}=\dfrac{z}{10}=k$ (suppose); then $x=2k$, $y=6k$ and $z=10k$.

Substituting in (3), we get $4k+18k+50k=36$, or $72k=36$, or $k=\frac{1}{2}$, $\therefore x=2\times\frac{1}{2}=1$, $y=6\times\frac{1}{2}=3$ and $z=10\times\frac{1}{2}=5$.

Ex. 4. Solve $x+y+z=0$..................................(1).

$ax+by+cz=0$..................................(2).

$bx+cy+az=a^2+b^2+c^2-ab-ac-bc$...(3).

From (1) and (2), $\dfrac{x}{c-b}=\dfrac{y}{a-c}=\dfrac{z}{b-a}=k$ (suppose).

$\therefore$ $x=k(c-b)$, $y=k(a-c)$ and $z=k(b-a)$.

Substituting these values in (3), we have—

$bk(c-b)+ck(a-c)+ak(b-a)=a^2+b^2+c^2-ab-ac-bc.$

$\therefore k\{bc-b^2+ca-c^2+ab-a^2\}=a^2+b^2+c^2-ab-ac-bc.$

$\therefore k=-1.$ $\quad\therefore$ $x=-(c-b)=(b-c)$, $y=-(a-c)=(c-a)$ and $z=-(b-a)=(a-b)$.

Ex. 5. Solve $x+y+z=a+b+c$............(1).

$\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=3$.....................(2).

$ax+by+cz=a^2+b^2+c^2$... (3).

The equations (1) and (2) may be written thus :—

$(x-a)+(y-b)+(z-c)=0.$

$\dfrac{1}{a}(x-a)+\dfrac{1}{b}(y-b)+\dfrac{1}{c}(z-c)=0.$

Hence, $\dfrac{x-a}{\frac{1}{c}-\frac{1}{b}}=\dfrac{y-b}{\frac{1}{a}-\frac{1}{c}}=\dfrac{z-c}{\frac{1}{b}-\frac{1}{a}}$,

or $\dfrac{bc(x-a)}{b-c}=\dfrac{ac(y-b)}{c-a}=\dfrac{ab(z-c)}{a-b}=k$ (suppose)

Then $x-a=\dfrac{k(b-c)}{bc}$ or $x=a+\dfrac{k(b-c)}{bc}$

Similarly $y=b+\dfrac{k(c-a)}{ac}$ and $z=c+\dfrac{k(a-b)}{ab}$

Substituting the values of x, y, z in (3), we have—

$$a^2+\frac{ak(b-c)}{bc}+b^2+\frac{bk(c-a)}{ac}+c^2+\frac{ck(a-b)}{ab}=a^2+b^2+c^2$$

$$\therefore\ k\left\{\frac{a(b-c)}{bc}+\frac{b(c-a)}{ac}+\frac{c(a-b)}{ab}\right\}=0$$

$\therefore\ k=0.$ $\qquad\therefore\ x=a,\ y=b$ and $z=c$.

EXERCISE 63.

Solve the following equations :—

1. $4x+3y-24=0$ and $5x-2y-7=0$
2. $x+11y-67=0$ and $10x+2y-130=0$
3. $11x-7y-41=0$ and $7x-5y-19=0$
4. $\dfrac{x}{2}+\dfrac{y}{5}-11=0$ and $\dfrac{x}{7}+\dfrac{y}{4}-7=0$.
5. $\dfrac{12}{x}-\dfrac{4}{y}-2=0$ and $\dfrac{15}{x}-\dfrac{8}{y}-1=0$.
6. $ax+y-p=0$ and $x-ay-q=0$.
7. $5x+6y+8z=0$, $3x+4y+6z=0$ and $x+5y+16z=3$.
8. $4x-4y+2z=0$, $2x-10y+7z=0$ and $3x+4y+5z=35$.
9. $x+y+z=0$, $ax+by+cz=0$ and $a^2x+b^2y+c^2z=(b-c)\times(c-a)(a-b)$.
10. $x+y+z=0$, $ax+by+cz=0$ and $bcx+cay+abz=(a-b)\times(b-c)(a-c)$.

11. $ax+by+cz=0$, $bcx+cay+abz=0$ and $x+y+z=1$.

12. $(b-c)x+(c-a)y+(a-b)z=0$, $ax+(a-c)y-cz=0$ and $x+y+z=a+b+c$.

13. $x+y+z=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, $a^2x+b^2y+c^2z=a+b+c$ and $a^3x+b^3y+c^3z=a^2+b^2+c^2$.

14. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$, $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $ax+by+cz=a^2+b^2+c^2$.

15. $x+y+z=0$, $bcx+acy+abz=1$ and $(b+c)x+(a+c)y+(a+b)z=0$.

16. $x+y+z=a+b+c$, $bx+cy+az=a^2+b^2+c^2$ and $cx+ay+bz=a^2+b^2+c^2$.

17. $x+y+z=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. $ax+by+cz=3$ and $a^2x+b^2y+c^2z=a+b+c$.

18. $x+y+z=a+b+c$, $bx+cy+az=ab+ac+bc$ and $cx+ay+bz=ab+ac+bc$.

19. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$, $ax+by+cz=a^2+b^2+c^2$ and $\frac{x-a}{b+c}+\frac{y-b}{c+a}+\frac{z-c}{a+b}=0$.

20. $x+y+z=0$, $\frac{x}{a+b}+\frac{y}{a+c}+\frac{z}{b+c}=0$ and $\frac{x}{a-b}+\frac{y}{c-a}+\frac{z}{b-c}=2(a+b+c)$.

148. Miscellaneous Examples.

1. Solve $x+y=8$, $xy=15$.

We know $(x-y)^2=(x+y)^2-4xy$.

Hence $(x-y)^2=64-60=4$.

$\therefore x-y=\pm2$ and $x+y=8$. $\quad\therefore 2x=10$ or 6.

$\therefore x=5$ or 3 and $y=3$ or 5.

If $x-y$ and xy be given, we can find the value of $x+y$ from $(x+y)^2=(x-y)^2+4xy$.

2. Solve $x^3+y^3=35$ and $x+y=5$.

$x^3+y^3=(x+y)(x^2-xy+y^2)$.

Hence $35=5(x^2-xy+y^2)$. $\therefore\ x^2-xy+y^2=7$.........(1).

Since $x+y=5$. $\therefore\ x^2+2xy+y^2=25$.................(2).

Subtracting (1) from (2), $3xy=18$. $\therefore\ xy=6$ and $x+y=5$

Hence x and y can be found as in Example 1.

3. Solve $xy=a^2$; $yz=b^2$ and $zx=c^2$.

Multiplying the three equations, $xy\times yz\times zx=a^2b^2c^2$, or $x^2y^2z^2=a^2b^2c^2$. $\therefore\ xyz=\pm abc$.

Hence $x=\dfrac{xyz}{yz}=\dfrac{\pm abc}{b^2}=\pm\dfrac{ac}{b}$, $y=\dfrac{xyz}{xz}=\dfrac{\pm abc}{c^2}=\pm\dfrac{ab}{c}$ and

$z=\dfrac{xyz}{xy}=\pm\dfrac{abc}{a^2}=\pm\dfrac{bc}{a}$.

4. Solve $x(x+y+z)=a^2-yz$; $y(x+y+z)=b^2-xz$ and $z(x+y+z)=c^2-xy$.

We have $x(x+y+z)+yz=a^2$. $\therefore\ x(x+y)+z(x+y)=a^2$.

$\therefore\ (x+y)(x+z)=a^2$.........(1).

Similarly from the other two equations we can get $(y+x)\times(y+z)=b^2$......(2) and $(z+x)(z+y)=c^2$.........(3).

Multiplying (1), (2) and (3), $(x+y)(x+z)(y+z)(y+x)\times(z+x)(z+y)=a^2b^2c^2$. $\therefore\ (x+y)^2(y+z)^2(z+x)^2=a^2b^2c^2$.

$\therefore\ (x+y)(y+z)(z+x)=\pm abc$............(4).

Dividing (4) by (1), (2) and (3) successively, we have

$y+z=\pm\dfrac{abc}{a^2}=\pm\dfrac{bc}{a}$. $z+x=\pm\dfrac{abc}{b^2}=\pm\dfrac{ac}{b}$ and $x+y=\pm\dfrac{abc}{c^2}$

$=\pm\dfrac{ab}{c}$.

From these three equations, the values of x, y and z can be found.

5 Solve $xy=\dfrac{1}{a}(x+y)$, $yz=\dfrac{1}{c}(y+z)$, $zx=\dfrac{1}{b}(z+x)$

From (1), $\dfrac{x+y}{xy}=a$, or $\dfrac{1}{x}+\dfrac{1}{y}=a$..............$(a)$

From (2), $\dfrac{y+z}{yz}=c$, or $\dfrac{1}{y}+\dfrac{1}{z}=c$..................($\beta$)

From (3), $\dfrac{z+x}{zx}=b$, or $\dfrac{1}{z}+\dfrac{1}{x}=b$..................($\gamma$)

$\dfrac{(\alpha)+(\beta)+(\gamma)}{2}=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{a+b+c}{2}$..................($\delta$)

Subtract (α), (β) and (γ) successively from (δ).

Then $\dfrac{1}{z}=\dfrac{b+c-a}{2}$, $\dfrac{1}{x}=\dfrac{a+b-c}{2}$ and $\dfrac{1}{y}=\dfrac{a+c-b}{2}$.

6. Solve $x(x+y+z)=a^2$............(1).

$y(x+y+z)=b^2$............(2).

$z(x+y+z)=c^2$............(3)

Adding (1), (2) and (3), we have $(x+y+z)^2=a^2+b^2+c^2$.

Hence $x+y+z=\pm\sqrt{(a^2+b^2+c^2)}$..................(4).

Dividing (1), (2) and (3) by (4) successively we get—

$$x=\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}},\ y=\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}},\ z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}.$$

7. Solve $x(y+z)=14$............(1).

$y(x+z)=18$............(2).

$z(x+y)=20$............(3).

Adding (1), (2) and (3), we have $2(xy+yz+zx)=52$.

Hence $xy+yz+zx=26$........................(4).

Subtracting (1), (2) and (3) successively from (4) we get $yz=12$, $xz=8$ and $xy=6$. From these the values of x, y and z can be got as in Example 3.

8. Solve $xy+3(x+y)=11$............(1)

$yz+3(y+z)=21$............(2).

$zx+3(z+x)=15$............(3).

From (1), $xy+3(x+y)+9=11+9=20$.

$\therefore\ (x+3)(y+3)=20$.................................. (A)

From (2), $yz+3(y+z)+9=21+9=30$.

$\therefore\ (y+3)(z+3)=30$ .... (B)

From (3), $zx+3(z+x)+9=15+9=24$.

$\therefore\ (z+3)(x+3)=24$ .... (C)

$(A)\times(B)\times(C)$ is $(x+3)^2(y+3)^2(z+3)^2=20\times30\times24$.

$\therefore\ (x+3)(y+3)(z+3)=10\times12=120$ .... (D)

$\frac{D}{A}$ is $z+3=\frac{120}{20}=6$. $\therefore\ z=3$.

$\frac{D}{B}$ is $x+3=\frac{120}{30}=4$. $\therefore\ x=1$.

$\frac{D}{C}$ is $y+3=\frac{120}{24}=5$. $\therefore\ y=2$.

EXERCISE 64.

Solve the following equations —

1. $x+y=1$ and $x^3+y^3=1-3b+3b^2$.

2. $x+y=\frac{3}{\sqrt{2}}$ and $x^3+y^3=\frac{9}{2\sqrt{2}}$.

3. $x+y=7$ and $x^2+y^2=25$. 4. $x-y=b$ and $\frac{x}{y}=\frac{5}{4}$.

5. $x+y=4$ and $\frac{1}{x}+\frac{1}{y}=1$. 6. $x-y=2$ and $x^3-y^3=8$.

7. $x+y=12$, $x^2+y^2=74$. 8. $x+y=11$, $x^3+y^3=1001$.

9. $abxy=ax+by$, $acxz=ax+cz$ and $bcyz=by+cz$.

10. $x(x+y+z)=6$, $y(x+y+z)=12$ and $z(x+y+z)=18$.

11. $(x^2+y^2)(y^2+z^2)=65$, $(y^2+z^2)(z^2+x^2)=130$ and $(z^2+x^2)(x^2+y^2)=50$.

12. $x(px+qy+rz)=p(p^2+q^2+r^2)$, $y(px+qy+rz)=q(p^2+q^2+r^2)$ and $z(px+qy+rz=r(p^2+q^2+r^2)$.

13. $x(y+z)=a$, $y(x+z)=b$ and $z(x+y)=c$.

14. $xy+a(x+y)=yz+a(y+z)=zx+a(z+x)=3a^2$.

15. $xy+x+y=11$, $yz+y+z=19$ and $zx+z+x=14$.

16. $ax+by+cz=mxyz$, $bx+cy+az=nxyz$ and $cx+ay+bz=lxyz$.

17. $x+y+z=\dfrac{a^2-yz}{x}=\dfrac{b^2-zx}{y}=\dfrac{c^2-xy}{z}$.

18. $\dfrac{y^2+z^2}{y^2z^2}=a^2$, $\dfrac{z^2+x^2}{z^2x^2}=b^2$ and $\dfrac{x^2+y^2}{x^2y^2}=c^2$.

149. Method of Undetermined Multipliers.

Take the equations $ax+by+cz=d$............(1).

$$a_1x+b_1y+c_1z=d_1\text{......(2)}.$$
$$a_2x+b_2y+c_2z=d_2\text{.......(3)}.$$

Multiplying (1) by l, (2) by m, (3) by n and adding, we get $(al+a_1m+a_2n)x+(bl+b_1m+b_2n)y+(cl+c_1m+c_2n)z=dl+d_1m+d_2n$.......................................(4).

Since l, m and n, are quite arbitrary, we may assume them to be such that the co-efficients of y and z shall vanish; hence

$bl+b_1m+b_2n=0$............(5)

$cl+c_1m+c_2n=0$............(6)

and $(al+a_1m+a_2n)x=dl+d_1m+d_2n$............(7).

From (5) and (6) by cross multiplication, $\dfrac{l}{b_1c_2-b_2c_1}$

$$=\frac{m}{b_2c-bc_2}=\frac{n}{bc_1-b_1c}=k \text{ (suppose)}.$$

$$\therefore\ l=(b_1c_2-b_2c_1)k,\ m=(b_2c-bc_2)k,\ n=(bc_1-b_1c)k.$$

Substituting these values of l, m, n, in (7), we have—

$$\{a(b_1c_2-b_2c_1)+a_1(b_2c-bc_2)+a_2(bc_1-b_1c)\}x$$
$$=d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c).$$

$$\therefore\ x=\frac{d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)}{a(b_1c_2-b_2c_1)+a_1(b_2c-bc_2)+a_2(bc_1-b_1c)}.$$

In the same way we can find the value of y by supposing the co-efficients of x and z in (4) to vanish, and the value of z by supposing the co-efficients of x and y in (4) to vanish.

Ex. *Solve the equations* $x+2y+z=8 \ldots\ldots\ldots\ldots(1).$
$2x-y+3z=9 \ldots\ldots\ldots\ldots(2).$
$3x+y-z=2 \ldots\ldots\ldots\ldots(3).$

Multiplying (1), (2) and (3) by l, m and n, respectively, and adding, we have—

$$(l+2m+3n)x+(2l-m+n)y+(l+3m-n)z=8l+9m+2n \ldots(4).$$

Equating the co-efficients of y and z to zero, we have—

$(l+2m+3n)x=8l+9m+2n \ldots\ldots\ldots\ldots\ldots\ldots(5).$

$2l-m+n=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$

$l+3m-n=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$

From (6) and (7) by cross multiplication, $\dfrac{l}{-2}=\dfrac{m}{3}=\dfrac{n}{7}=k$ (suppose).

Substituting $-2k$, $3k$ and $7k$ for l, m and n in (5), we have $(-2+6+21)x=-16+27+14.$

$\therefore\ 25x=25.\quad \therefore\ x=1.$ Similarly we can find $y=2$ and $z=3$.

Note.—In Practice, this method is not simpler than the one given in Art. (146) when the co-efficients are numerical.

CHAPTER XXV.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

150. The *number* of unknown quantities which enter a problem and the *number* of independent conditions connecting them with the known quantities must be the *same.*

The unknown quantities are usually denoted by x, y and z.

Ex. 1. There is a fraction, such that if its numerator be increased by 1, its value is $\frac{3}{4}$; and if its denominator be increased by 2, its value is $\frac{1}{2}$. What is the fraction?

Let the fraction be $\dfrac{x}{y}$.

If we increase the numerator by 1, the fraction becomes $\dfrac{x+1}{y}$.

,, denominator by 2, the fraction becomes $\dfrac{x}{y+2}$.

By the conditions of the problem, $\dfrac{x+1}{y}=\frac{3}{4}$ and $\dfrac{x}{y+2}=\frac{1}{2}$.

$\therefore\ 4x+4=3y$ and $2x=y+2$. $\therefore\ 4x=2y+4$.

$\therefore\ 2y+4+4=3y$. $\therefore\ y=8$ and $x=5$.

Hence $\frac{5}{8}$ is the required fraction.

Ex. 2. There is a number consisting of two digits; the number is equal to seven times the sum of its digits, and if 27 be subtracted from the number, the digits interchange their places; find the number.

Let $x=$ the digit in the units' place,

and $y=$ the digit in the tens' place.

Then $10y+x=$ the number,

and $10x+y=$ the number when the digits are *interchanged*

By the conditions of the problem, $10y+x=7(x+y)$...(1)

and $10y+x-27=10x+y$...(2).

From (1), $3y-6x=0$. $\therefore 3y=6x$. $\therefore y=2x$.

Substituting $2x$ for y in (2), we get $20x+x-27=10x+2x$.

$\therefore 9x=27$. $\therefore x=3$ and $y=6$. Hence the number is 63.

Ex. 3. There is a number of three digits; the sum of the digits is 9, the digit in the units' place is twice the digit in the hundreds' place, and if 198 be added to the number, the digits are reversed; find the number.

Let x, y and z be the digits in the hundreds', tens' and units' places respectively; then $100x+10y+z$ is the number,

and $100z+10y+x$ is the number formed when the digits are *reversed*. By the question, $x+y+z=9$......................(1)

$z=2x$...(2) and $100x+10y+z+198=100z+10y+x$......(3).

From (3), $99x-99z=-198$. $\therefore x-z=-2$

but $z=2x$. $\therefore x-2x=-2$. $\therefore x=2$ and $z=4$.

From (1), $y=9-6=3$. Hence the required number is 234.

Ex. 4. A and B working together can do a certain work in 4 days. A and C can do it in $3\frac{3}{5}$ days, while B and C can do it in $5\frac{1}{7}$ days. How many days will A, B and C each take to do it alone?

Let x = the number of *days* A takes to do the work alone;

y = the number of *days* B takes to do the work alone;

and z = the number of *days* C takes to do the work alone.

Hence in one day A does $\frac{1}{x}$ of the work, B does $\frac{1}{y}$ of the work and C does $\frac{1}{z}$ of the work.

$\therefore$ in one day A and B together do $\left(\frac{1}{x}+\frac{1}{y}\right)$ of the work.

But as they take 4 days to do the whole work, in one day they must do $\frac{1}{4}$ of the work.

Therefore $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$...............(1).

Similarly $\frac{1}{x}+\frac{1}{z}=\frac{1}{3\frac{3}{5}}=\frac{5}{18}$........(2).

and $\frac{1}{y}+\frac{1}{z}=\frac{1}{5\frac{1}{7}}=\frac{7}{36}$..............(3).

From (2) and (3), $\left.\begin{aligned}\frac{1}{x}-\frac{1}{y}&=\frac{1}{12}\\ \text{and from (1), }\frac{1}{x}+\frac{1}{y}&=\frac{1}{4}\end{aligned}\right\}$ $\therefore$ $\frac{1}{x}=\frac{1}{6}$ $\therefore x=6$ days,

and $\frac{1}{y}=\frac{1}{12}$. $\therefore y=12$ days,

and from (2), $\frac{1}{z}=\frac{5}{18}-\frac{1}{6}=\frac{1}{9}$. $\therefore z=9$ days.

Ex. 5. A man has to travel a certain distance; when he has travelled 20 miles, he increases his speed 1 mile per hour; if he had travelled with his increased speed during the whole of his journey, he would have arrived 40 minutes earlier; but if he had kept on at his first rate, he would have arrived 20 minutes later; how far had he to travel?

Let x = the distance of the journey in *miles*,

and y = rate per hour in *miles*.

At first he takes $\left(\frac{20}{y}+\frac{x-20}{y+1}\right)$ hours. When he travels the whole distance at $y+1$ miles per hour he takes $\frac{x}{y+1}$ hours and at y miles per hour, he takes $\frac{x}{y}$ hours.

By the question. $\frac{20}{y}+\frac{x-20}{y+1}=\frac{x}{y+1}+\frac{2}{3}$.........(1).

and $\frac{20}{y}+\frac{x-20}{y+1}=\frac{x}{y}-\frac{1}{3}$..............................(2).

$\therefore \frac{x}{y+1}+\frac{2}{3}=\frac{x}{y}-\frac{1}{3}$. $\therefore \frac{x}{y}-\frac{x}{y+1}=1$ [From (1) and (2)].

$\therefore \frac{x}{y(y+1)}=1$. $\therefore x=y(y+1)$.

Substituting $y(y+1)$ for x in (1), we get—

$$\frac{20}{y}+\frac{y(y+1)-20}{y+1}=\frac{y(y+1)}{y+1}+\frac{2}{3}.$$

$$\therefore \frac{20}{y}+y-\frac{20}{y+1}=y+\frac{2}{3}. \quad \therefore \frac{20}{y}-\frac{20}{y+1}=\frac{2}{3}.$$

$\therefore \dfrac{20}{y(y+1)}=\dfrac{2}{3}$. $\therefore$ $y(y+1)=\dfrac{3}{2}\times 20=30$.

$\therefore$ $x=30$, since $x=y(y+1)$. $\therefore$ the distance $=30$ miles.

Ex. 6. A pound of tea and three pounds of sugar cost six shillings, but if sugar were to rise 50 per cent., and tea 10 per cent., they would cost seven shillings. Find the price of tea and sugar.

Let $x=$ the price of a pound of tea in *shillings*;

and $y=$ the price of a pound of sugar in *shillings*.

When the price of tea *rises* 10 per cent., the price of a pound $=x+\frac{1}{10}x$ or $\frac{11}{10}x$ shillings; and when the price of sugar *rises* 50 per cent., the price of a pound $=y+\frac{1}{2}y$ or $\frac{3}{2}y$ shillings.

By the question, $x+3y=6$ . . . . (1)

and $\frac{11}{10}x+3\times\frac{3}{2}y=7$ . . . . (2).

From (2), $11x+45y=70$. From (1), $11x+33y=66$.

$\therefore$ $12y=4$. $\therefore$ $y=\frac{1}{3}$ and $x=6-1=5$. Hence the price of a pound of tea $=5s.$, and that of a pound of sugar $=\frac{1}{3}s.$ or $4d.$

Ex. 7. The dimensions of a rectangular court are such that if the length were increased by 3 feet and the breadth diminished by the same, its area would be diminished by 18 square feet; and if its length were increased by 1 foot and the breadth increased by the same, its area would be increased by 18 square feet. Find the dimensions.

Let $x=$ the length in *feet* and $y=$ the breadth in *feet*. Then $xy=$ the area in *square feet*.

By the question, $(x+3)(y-3)=xy-18$ . . . (1) and $(x+1)\times(y+1)=xy+18$ . . . . (2).

From (1), $3(y-x)-9=-18$. $\therefore$ $y-x=-3$.

From (2), $x+y+1=18$. $\therefore$ $x+y=17$. $\therefore$ $x=10$ and $y=7$.

$\therefore$ length $=10$ feet and breadth $=7$ feet.

EXERCISE 65.

1. Find two numbers such that one-half of the first and one-third of the second is 14, and one-third of the first and one-half of the second is 11.

2. Eight years ago A was 5 times as old as B, and in 2 years he will be 3 times as old; find their present ages.

3. Find the fraction which becomes $\frac{1}{2}$ when 8 is added to its numerator, and $\frac{1}{10}$ when 8 is subtracted from its denominator.

4. A number consisting of 2 digits is such that, if it is divided by the sum of its digits, the quotient is 8 and the remainder is 2; but if the number with the digits reversed be divided by the sum of the digits the quotient is 2 and the remainder is 8; find the number.

5. A number of three digits is such that the sum of its digits is 16, the digit in the units' place is twice the digit in the hundreds' place, and if 297 be added to the number, the digits are reversed; find the number.

6. A person buys 8 lbs. of tea and 5 lbs. of sugar for 19*s.* 11*d.*; and at another time he buys 5 lbs. of tea and 8 lbs. of sugar for 13*s.* 8*d.*, the prices being the same as before; find the price of each.

7. In a mile race A can beat B by 100 yards, and C by 200 yards; by how much can B beat C in a mile?

8. There is a number consisting of three digits, and is such that the sum of the first and last digits exceeds twice the middle one by unity; when the digits are inverted the original number is increased by 297, and the sum of the digits is 16; find the number.

9. Two trains start from two stations, and each proceeds at a uniform rate towards the other station; when they meet it is found that one has travelled 108 miles more than the other, and that if they continue to travel at the same rate they will finish the journey in 9 and 16 hours respectively; find the rates of the trains and the distance between the two stations.

10. A person wishing to relieve a certain number of beggars finds that if he give them 2*s.* each, he will not have money enough by 3*s.*; but if he give them 1*s.* 6*d.* each, he will have 4*s.* 6*d.* to spare. What money had he in his pocket and how many beggars did he relieve?

11. A certain sum of money is to be divided among a certain number of men; if there were three men less, each man would have £150 more; but if there were 6 men more, each man would have £120 less; find the sum of money and the number of men.

12. A person has two horses, and a saddle worth £10; if the saddle be put on the first horse, his value becomes double that of the second; but if the saddle be put on the second horse, his value will be less than that of the first by £13. Find the value of each horse.

13. A person walks from A to B, a distance of $7\frac{1}{2}$ miles, in 2 hours $17\frac{1}{2}$ minutes and returns in 2 hours 20 minutes. His rates of walking up-hill, down-hill, and on level road being 3, $3\frac{1}{2}$ and $3\frac{1}{4}$ miles respectively; find the length of the level road between A and B.

14. There are two fractions such that the fraction formed with the sum of their numerators for numerator and the sum of their denominators for denominator is $\frac{8}{9}$ of the greater; and the fraction similarly formed with the difference of the numerators and denominators is $\frac{1}{2}$; also the sum of the numerators is twice the difference of the denominators. Find the fractions.

15. If A and B together can do a piece of work in a days, A and C together can do the same in b days, B and C together in c days; find the time in which each can perform it separately.

16. The fore-wheel of a coach makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased 1 yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same distance. Find the circumference of each wheel.

17. A fish was caught whose tail weighed 9 lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh?

18. Two trains 140 feet and 80 feet long pass each other in opposite directions in 2 seconds; had they been going in the same direction the faster would have passed the other in 10 seconds; what was the rate of each in miles per hour?

19. A cistern is filled in 25 minutes by 3 pipes, one of which conveys 8 gallons more, and another 7 gallons less than the third, every three minutes. The cistern holds 1,050 gallons. How much flows through each pipe in a minute?

20. A and B run a mile. First A gives B a start of 1 minute 30 seconds, and is beaten by 88 yards. At the second heat A gives B a start of 80 yards and beats him by 30 seconds. Find the rates of A and B per hour.

21. Two travellers A and B set out at the same time from two places P and Q respectively and travel so as to meet. When they meet, it is found that A has travelled 30 miles more than B, and that A will reach Q in 4 days, and B will reach P in 9 days after the meeting. Find the distance between P and Q.

22. A boy when he started for school, a distance of 4 miles, found that if he proceeded at his walking pace, he would reach school just in time. But at the end of the first mile, he turned back and ran home to bring a book, and thence back to the end of the first mile, increasing his rate by 1 mile an hour. If by walking the remaining 3 miles at twice his walking pace, he reached the school exactly in time, find his walking pace per hour.

23. Two post runners start at the same time from two towns and each proceeds at a uniform rate towards the other. They meet at a point 3 miles nearer one town than the other and they expect to finish the journey in 2 hours 30 minutes, and 3 hours 36 minutes respectively. Find the distance between the towns and the rates of the runners.

24. A and B set out together to walk round a field one mile in circumference. When A, who is the quicker walker has passed B twice, B turns and going in the opposite direction meets A at the starting point. The whole time since they started is 3 hours, and A has walked $2\frac{1}{4}$ miles since they started more than B. Find their rates of walking.

25. A mail cart was travelling from A to B at the rate of 10 miles an hour, and when only 7 miles from B was met by a man who started from B at the same time that the cart left A. The cart then went on to B where remaining 1 hour and 12 minutes, it returned overtaking the man 6 miles from A. Find the distance from A to B and also the rate of the man.

26. A rectangular Court having been measured it was observed that, if it were 5 feet broader and 4 feet longer, it would contain 116 square feet more; but, if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more. Find its dimensions.

27. Find a number of three digits, the last two alike, such that the number formed by the digits inverted may exceed twice the original number by 42 and also the number formed by putting the single figure in the midst by 27.

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28. A sum of money is divided equally among a certain number of persons; if there had been four more, each would have received a shilling less than he did; if there had been five fewer, each would have received 2 shillings more than he did; find the number of persons and what each received.

29. Two plugs are opened in the bottom of a cistern containing 192 gallons of water; after three hours, one of the plugs becomes stopped, and the cistern is emptied by the other in eleven hours; had six hours elapsed before the stoppage, it would have required only six hours more to empty the cistern. How many gallons will each plug-hole discharge in an hour, supposing the discharge uniform?

30. If there were no accidents it would take half as long to travel the distance from A to B by railroad as by coach; but three hours being allowed for accidental stoppages by the former, the coach will travel the distance all but fifteen miles in the same time; if the distance were two-thirds as great as it is, and the same time allowed for railway stoppages, the coach would take exactly the same time. Required the distance.

31. A and B run a mile. First A gives B a start of 44 yards and beats him by 51 seconds; at the second heat A gives B a start of 1 minute 15 seconds, and is beaten by 88 yards. Find the times in which A and B can run a mile separately.

32. A and B start together from the foot of a mountain to go to the summit. A would reach the summit half an hour before B, but missing his way goes a mile and back again needlessly, during which, he walks at twice his former pace, and reaches the top six minutes before B. C starts twenty minutes after A and B, and walking at the rate of two and one-seventh miles per hour, arrives at the submit ten minutes after B. Find the rates of walking of A and B, and the distance from the foot to the top of the mountain.

33. A railway train after travelling for one hour meets with an accident which delays it one hour, after which, it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the line, and the original rate of the train.

34. A railway train running from London to Cambridge meets on the way with an accident, which causes it to diminish

its speed to $\frac{1}{n}$th of what it was before, and it is in consequence a hours late. If the accident had happened b miles nearer Cambridge the train would have been c hours late. Find the rate of the train before the accident occurred.

35. A shop-keeper on account of bad book-keeping, knows neither the weight nor the prime cost of a certain article which he purchased. He only recollects that if he had sold the whole at 30*s*. per lb. he would have gained £5 by it, and if he had sold it at 22*s*. per lb. he would have lost £15 by it. What was the weight and prime cost of the article?

36. The rent of a farm is paid in certain fixed numbers of quarters of wheat and barley; when wheat is at 55*s*. and barley at 33*s*. per quarter, the portions of rent by wheat and barley are equal to one another, but when wheat is at 65*s*. and barley at 41*s*. per quarter, the rent is increased by £7. What is the corn rent?

37. A train 60 yards long passed another train 72 yards long, which was travelling in the same direction on a parallel line of rails in 12 seconds. Had the slower train been travelling half as fast again, it would have been passed in 24 seconds. Find the rates at which the trains were travelling.

38. A and B run a race round a two-mile course. In the first heat B reaches the winning post 2 minutes before A. In the second heat A increases his speed by 2 miles an hour, and B diminishes his by the same quantity; and A then arrives at the winning post 2 minutes before B. Find at what rate each ran in the first heat.

39. A and B run a mile. At the first heat A gives B a start of 20 yards and beats him by 30 seconds. At the second heat A gives B a start of 32 seconds, and beats him by $9\frac{5}{11}$ yards. Find the rate per hour at which A runs.

40. A railway train after travelling an hour is detained 15 minutes, after which, it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 3 minutes sooner than it did. Find the original rate of the train and the distance travelled.

41. The time which an express train takes to travel a journey of 120 miles is to that taken by an ordinary train as 9

to 14. The ordinary train loses as much time in stoppages as it would take to travel 20 miles without stopping. The express train only loses half as much time in stoppages as the ordinary train, and it also travels 15 miles an hour quicker. Find the rate of each train.

42. Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails; when they move in opposite directions, they are observed to pass each other in one second and a half; but when they move in the same direction the faster train is observed to pass the other in six seconds; find the rate at which each train moves.

43. A general finds that his cavalry with half his artillery and infantry together, or his artillery with one-third of his cavalry and infantry together, or his infantry with one-fourth of his cavalry and artillery together, make up the same number of men; *viz.*, 5,950; how many men were there in each arm of the service?

44. There is a certain number of three digits which is equal to 48 times the sum of its digits, and if 198 be subtracted from the number, the digits will be reversed; also the sum of the extreme digits is equal to twice the middle digit; find the number.

45. Three cases of goods cost Rs. 4,000; they are sold again at a profit of 2, 3, 4 per cent. respectively, and the whole profit is 3 per cent. on the total cost; if the first and second cases had been sold for Rs. 5 more each, and the third for what it had cost, the profit would have been 2 per cent; what was the cost of each case?

46. In a school consisting of 3 classes, there were in the second class 5 per cent., and in the third class 10 per cent., more than in the first class. In an examination, each boy in the first class occupied thrice, and in the second class twice, as much of the examiner's time as each boy in the third. The examination then lasted 31 hours. In the next year the first class had doubled in number, but each boy only required $\frac{2}{3}$ of his former time, there were 10 boys more in the second class, and each boy in the third class occupied $\frac{1}{11}$ hour more than he did before. The examination now lasted 43 hours. How many boys were there in the school at first?

47. A person starts from A to walk to B, and after he has gone 16 miles, another starts from B for A and walks at double

the rate of the former. Thirty-two minutes after they meet, the slower of the two halts for two hours, and then does the remainder of the distance in 4 hours and 48 minutes. The other proceeds without stopping to A and immediately returns to B which he reaches 4 hours 24 minutes after the other. Find the distance between A and B, and the distance from B of the halting place.

48. A and B start from opposite corners of a square and run round in the same direction. If B stops for a certain time at every corner he will be caught by A when he is just commencing his $(n+1)$th circuit; but if B runs continuously and A stops the same time at each corner, A will be caught when he is just commencing his $(m+1)$th circuit. Shew that A's velocity : B's velocity as $4+n^{-1} : 4+m^{-1}$.

CHAPTER XXVI.

QUADRATIC EQUATIONS.

151. Quadratic Equations are those into which the second power of the unknown quantity enters, with or without the first power.

If the second power of the unknown quantity alone enters, such equations are called *Pure Quadratic Equations*; thus $x^2=25$ and $x^2-b=7$, are pure quadratic equations.

If the first power as well as the second power of the unknown quantity be involved, such equations are called *Adfected Quadratic Equations*: thus $x^2+2x=3$ and $ax^2+bx=c$, are adfected quadratic equations.

152. Pure Quadratic Equations are solved in the same manner, in every respect, as simple equations, except that, at the conclusion, the square root of each side of the equation has to be taken. The signs + and − are prefixed to the root, because the square root of a quantity may be either positive or negative.

Ex. 1. Solve $12x^2-16=284$.

$\therefore\ 12x^2=284+16=300.\ \ \therefore\ x^2=\frac{300}{12}=25.$

$\therefore\ \pm x=\sqrt{25}=\pm 5.$

i.e., $+x=+5, +x=-5, -x=+5$ and $-x=-5$,

i.e., $x=+5$ or -5, *i.e.* $x=\pm 5$.

Hence it follows that when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of the right hand side.

Ex. 2. Solve $2bx^2+a-4=cx^2-5+d-bx^2$.

$\therefore\ 2bx^2-cx^2+bx^2=d-5-a+4;\ \therefore\ x^2(3b-c)=d-a-1$

$$\therefore\ x^2=\frac{d-a-1}{3b-c};\ \therefore\ x=\pm\sqrt{\left(\frac{d-a-1}{3b-c}\right)}.$$

Ex. 3. Solve $x+\sqrt{x^2-a^2}=\dfrac{2a^2}{\sqrt{x^2-a^2}}$,

or $x\sqrt{x^2-a^2}+x^2-a^2=2a^2$, or $x\sqrt{x^2-a^2}=3a^2-x^2$.

Squaring, $x^2(x^2-a^2)=(3a^2-x^2)^2$,

or $x^4-a^2x^2=9a^4+x^4-6a^2x^2$, or $5a^2x^2=9a^4$, or $5x^2=9a^2$,

or $x^2=\dfrac{9a^2}{5}$. $\therefore\ x=\pm\dfrac{3a}{\sqrt{5}}$.

Ex. 4. Solve $\dfrac{b-\sqrt{b^2-x^2}}{b+\sqrt{b^2-x^2}}=a$.

$$\therefore \frac{b-\sqrt{b^2-x^2}+b+\sqrt{b^2-x^2}}{b-\sqrt{b^2-x^2}-b-\sqrt{b^2-x^2}}=\frac{a+1}{a-1}\ \text{(II of Art. 132)},$$

or $-\dfrac{b}{\sqrt{b^2-x^2}}=\dfrac{a+1}{a-1}$. $\therefore -\dfrac{\sqrt{b^2-x^2}}{b}=\dfrac{a-1}{a+1}$.

$$\therefore \frac{b^2-x^2}{b^2}=\left(\frac{a-1}{a+1}\right)^2. \quad \therefore 1-\frac{x^2}{b^2}=\left(\frac{a-1}{a+1}\right)^2.$$

$$\therefore \quad \frac{x^2}{b^2}=1-\left(\frac{a-1}{a+1}\right)^2=\frac{4a}{(a+1)^2}.$$

$$\therefore \quad x^2=\frac{4ab^2}{(a+1)^2}. \quad \therefore x=\pm\frac{2b\sqrt{a}}{(a+1)}.$$

EXERCISE 66.

Solve the following equations —

1. $(x+3)^2=6x+10$
2. $x(x+1)+(x-3)(x+2)=4$.
3. $\dfrac{2}{1+x}+\dfrac{2}{1-x}=5$
4. $(x+a)^2+(x-a)^2=3a^2$.
5. $\dfrac{x-1}{x}-\dfrac{3x}{x-1}=2$
6. $5x-\dfrac{12}{x}=0$.
7. $\dfrac{x+2}{2x+1}=\dfrac{x+3}{3x+1}$.
8. $\dfrac{x-2}{x+2}+\dfrac{x+2}{x-2}=2\dfrac{x+3}{x-3}$.
9. $\sqrt{x^2+16}+\sqrt{x^2+7}=9$.
10. $\sqrt{4x^2+9}-\sqrt{4x^2-7}=2$
11. $\sqrt{x^2+13}-\sqrt{x^2-11}=2$
12. $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}$.

13. $x+\sqrt{a^2+x^2}=\dfrac{na^2}{\sqrt{a^2+x^2}}$.

14. $\dfrac{x+3}{x+2}+\dfrac{x-3}{x-2}=\dfrac{2x-3}{x-1}$.

15. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\dfrac{x}{2a}$.

16. $\dfrac{3x+\sqrt{9x^2-4}}{3x-\sqrt{9x^2-4}}=4$. 17. $\dfrac{a}{x}+\dfrac{\sqrt{a^2-x^2}}{x}=\dfrac{x}{b}$.

18. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=4$. 19. $\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}=\dfrac{b}{c}$.

20. $\dfrac{\sqrt{a^2-x^2}-\sqrt{b^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}}=\dfrac{c}{d}$.

21. $\dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}}=\dfrac{a}{x^2}$.

22. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$.

23. $25x\{\sqrt{(1+x^2)}-x\}=3$.

24. $\sqrt{(a+x)}-\sqrt{(a-x)}=\sqrt{\left(\dfrac{a^2+2ab}{b+\frac{1}{2}a}\right)}$.

25. $\sqrt{\left(\dfrac{x+2}{x-2}\right)}+\sqrt{\left(\dfrac{x-2}{x+2}\right)}=4$.

153. Adfected Quadratic Equations.

The most general form of such equations is $ax^2+bx+c=0$ where a, b, c are known quantities, positive or negative, integral or fractional.

Dividing both sides of the equation by a, the co-efficient of x^2, we have $\dfrac{ax^2}{a}+\dfrac{bx}{a}+\dfrac{c}{a}=\dfrac{0}{a}$, or $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, or $x^2+\dfrac{b}{a}x$

$$=-\frac{c}{a}, \text{ or } x^2+2.\frac{b}{2a}x=-\frac{c}{a}.$$

To make the left side expression a perfect square, we must add $\left(\frac{b}{2a}\right)^2$ or $\frac{b^2}{4a^2}$. Adding $\frac{b^2}{4a^2}$ to both sides, we have

$x^2+2.\frac{b}{2a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$, or $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$.

Extracting the square root,

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \text{ or } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

Note.—The quantity to be added to $x^2+\frac{b}{a}x$ to make it a complete square is *the square of half the co-efficient of* x, i.e., $\left(\frac{b}{2a}\right)^2$.

From the preceding we derive the rule for the solution of an adfected quadratic equation :—

"*By transposition and reduction arrange the equation so that the terms involving the unknown quantity are alone on one side, and the co-efficient of* x^2 *is* $+1$; *add to both sides of the equation the square of half the co-efficient of* x, *and extract the square root of both sides.*"

Ex. 1. Solve $-3x^2+36x-105=0$.
Changing the signs, $3x^2-36x+105=0$.
Dividing by 3, $x^2-12x+35=0$ or $x^2-12x=-35$.
Adding to both sides the square of half the co-efficient of x, i.e., $\left(\frac{12}{2}\right)^2$, *i.e.*, 36, $x^2-12x+36=36-35=1$ or $(x-6)^2=1$.

Extracting the square root of both sides, we get $x-6=\pm1$.
$\therefore x=6+1$ or $6-1$, *i.e.*, 7 or 5.

We may make use of the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and get the values of x thus :—

In the equation $-3x^2+36x-105=0$, $a=-3$, $b=36$ and $c=-105$.

Hence putting these values for a, b, c in the formula, we get $x=\frac{-36\pm\sqrt{36^2-4\times(-3)(-105)}}{-6}=\frac{-36\pm\sqrt{1296-1260}}{-6}$

$=\frac{-36\pm\sqrt{36}}{-6}=\frac{-36\pm6}{-6}=\frac{-42}{-6}$ or $\frac{-30}{-6}$, i.e., 7 or 5.

Ex. 2. Solve $x^2+px+q=0$.

Transposing, $x^2+px=-q$.

Adding to both sides the square of half the co-efficient of x, i.e., $\left(\frac{p}{2}\right)^2$, i.e., $\frac{p^2}{4}$, we have $x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q$.

$$\text{or } \left(x+\frac{p}{2}\right)^2=\frac{p^2-4q}{4},$$

$$\therefore\ x+\frac{p}{2}=\pm\frac{\sqrt{p^2-4q}}{2}.\quad \therefore\ x=\frac{-p\pm\sqrt{p^2-4q}}{2}.$$

Applying the formula (A), and putting 1 for a, p for b and q for c, we get $x=\frac{-p\pm\sqrt{p^2-4q}}{2}$ (the same values as before).

Ex. 3. Solve $x^2-2cx=a^2$.

$$\therefore\ x^2-2cx+\left(\frac{2c}{2}\right)^2=a^2+\left(\frac{2c}{2}\right)^2$$

$$\text{or } x^2-2cx+c^2=a^2+c^2 \text{ or } (x-c)^2=a^2+c^2.$$

$$\therefore\ x-c=\pm\sqrt{a^2+c^2}.\quad \therefore\ x=c\pm\sqrt{a^2+c^2}.$$

Applying the formula (A), $x=\frac{2c\pm\sqrt{4c^2+4a^2}}{2}$

$$=c\pm\sqrt{c^2+a^2}.$$

Ex. 4. Solve $3x^2+2x=456$ or $3x^2+2x-456=0$.

Here $a=3$, $b=2$ and $c=-456$.

Hence by the formula (A), $x=\frac{-2\pm\sqrt{4+12\times456}}{6}$

$$=\frac{-2\pm74}{6}=12 \text{ or } -12\tfrac{2}{3}.$$

Ex. 5. Solve $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$.

$$\therefore\ 6\{x^2+(x+1)^2\}=13x(x+1).$$

$$\therefore\ 6x^2+6x^2+12x+6=13x^2+13x.\qquad \therefore\ -x^2-x+6=0,$$

$x^2+x-6=0$. Here $a=1$, $b=1$ and $c=-6$.

$$\therefore\ x=\frac{-1\pm\sqrt{1+4\times6}}{2}=\frac{-1\pm5}{2}=2 \text{ or } -3.$$

Ex. 6. Solve $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{1}{x-c}$.

Multiplying both sides by $(x-a)(x-b)(x-c)$, we have $(x-b)(x-c)+(x-a)(x-c)=(x-a)(x-b)$,
or $x^2-(b+c)x+bc+x^2-(a+c)x+ac=x^2-(a+b)x+ab$,
or $x^2-2cx+bc+ac-ab=0$.

$$\therefore\ x=\frac{2c\pm\sqrt{4c^2-4(bc+ac-ab)}}{2}\ \text{\{by formula (A)\}}$$

$$=c\pm\sqrt{c^2-bc-ac+ab}=c\pm\sqrt{(c-a)(c-b)}.$$

Ex 7. Solve $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}=\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}$.

Putting y for $\dfrac{x+1}{x-1}$ and b for $\dfrac{a+1}{a-1}$, we have $y+\dfrac{1}{y}=b+\dfrac{1}{b}$.

$$\therefore\ y^2-\left(b+\frac{1}{b}\right)y+1=0$$

$$\therefore y=\frac{b+\frac{1}{b}\pm\sqrt{\left(b+\frac{1}{b}\right)^2-4}}{2}=\frac{b+\frac{1}{b}\pm\left(b-\frac{1}{b}\right)}{2}$$

$$=b \text{ or } \frac{1}{b}.$$

Hence $\dfrac{x+1}{x-1}=\dfrac{a+1}{a-1}$ or $\dfrac{x+1}{x-1}=\dfrac{a-1}{a+1}$.

$$\therefore\ \frac{x+1+x-1}{x+1-x+1}=\frac{a+1+a-1}{a+1-a+1} \text{ or } \frac{x+1+x-1}{x+1-x+1}=\frac{a-1+a+1}{a-1-a-1}$$

$\therefore\ x=a$ or $x=-a$.

Ex. 8. Solve $(3x-1)(4x+1)=0$.

The equation $(3x-1)(4x+1)=0$ states that the product of the two factors $(3x-1)$ and $(4x+1)$ must $=0$; and it is required to find what values must be substituted for x to produce this result.

Now it is evident that if *one* only of the factors be made equal to 0, then the product of the two must $=0$, whatever the other factor be; hence, if we can find those values of x which

make each factor separately equal to 0, both of those values will be roots of the equation.

Therefore, to solve the equation,

(i) put $3x-1=0$. $\therefore$ $3x=1$. $\therefore$ $x=\frac{1}{3}$.

(ii) put $4x+1=0$. $\therefore$ $4x=-1$ $\therefore$ $x=-\frac{1}{4}$.

That is, $\frac{1}{3}$ and $-\frac{1}{4}$ are the two roots of the equation.

Generally, if an equation can be put in the form $(x-a)\times(x-b)=0$ the roots of the equation can be at once determined by putting each factor separately $=0$.

Ex. 9. Solve $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$.

Squaring both sides, $x+3+x+8+2\sqrt{(x+3)(x+8)}=25x$.

Transposing, $2\sqrt{(x+3)(x+8)}=23x-11$.

Squaring, $4(x+3)(x+8)=(23x-11)^2$

or $4x^2+44x+96=529x^2-506x+121$

or $525x^2-550x+25=0$ or $21x^2-22x+1=0$

or $(21x-1)(x-1)=0$.

$\therefore$ (i) $21x-1=0$. $\therefore$ $21x=1$. $\therefore$ $x=\frac{1}{21}$.

(ii) $x-1=0$. $\therefore$ $x=1$.

Therefore the roots of the equation are 1 and $\frac{1}{21}$.

Ex 10. Solve $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}$.

$$\therefore \frac{x-\sqrt{x+1}+x+\sqrt{x+1}}{x+\sqrt{x+1}-x+\sqrt{x+1}}=\frac{5+11}{11-5}.$$

$\therefore \dfrac{x}{\sqrt{x+1}}=\dfrac{8}{3}$. $\therefore$ $3x=8\sqrt{x+1}$.

$\therefore 9x^2=64(x+1)$. $\therefore$ $9x^2-64x-64=0$

or $(9x+8)(x-8)=0$. $\therefore$ $x=8$ or $-\dfrac{8}{9}$.

Applying the formula (A), we have—

$$x=\frac{64\pm\sqrt{(64)^2+36\times 64}}{18}=\frac{64\pm 80}{18}=8 \text{ or } -\frac{8}{9}.$$

Ex. 11. Solve $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2.$

Transposing, $(7-4\sqrt{3})x^2+(2-\sqrt{3})x-2=0.$

$$7-4\sqrt{3}=(2-\sqrt{3})^2.$$

Applying the formula (A) of Art. 152, we have—

$$x=\frac{-(2-\sqrt{3})\pm\sqrt{\{(2-\sqrt{3})^2+8(2-\sqrt{3})^2\}}}{2(2-\sqrt{3})^2}$$

$$=\frac{-(2-\sqrt{3})\pm3(2-\sqrt{3})}{2(2-\sqrt{3})^2}=\frac{1}{2-\sqrt{3}}\text{ or }-\frac{2}{2-\sqrt{3}}=2+\sqrt{3}\text{ or }$$

$-2(2+\sqrt{3}).$

EXERCISE 67.

Solve the following equations :—

1. $x^2-2x=8.$ 2. $x^2-14x=120.$ 3. $x^2+32x=320.$

4. $x^2+7x=8.$ 5. $x^2+19x=20.$ 6. $x^2+111x=3400.$

7. $x=\frac{5}{3}+\frac{1}{12}x^2.$ 8. $2x=4+\frac{6}{x}.$ 9. $2x^2+1=11(x+2).$

10. $\frac{3}{4}(x^2-3)=\frac{1}{8}(x-3).$ 11. $x-\frac{x^3-8}{x^2-5}=2.$

12. $\frac{1}{3}+\frac{1}{x+3}+\frac{1}{2x+3}=0.$ 13. $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}.$

14. $\frac{x+22}{3}-\frac{4}{x}=\frac{9x-6}{2}.$ 15. $\frac{12}{5-x}+\frac{4}{4-x}=\frac{32}{x+2}.$

16. $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35}.$ 17. $\frac{5x}{x+4}-\frac{3x-2}{2x-3}=2.$

18. $\frac{3x-7}{x}+\frac{4x-10}{x+5}=3\frac{1}{2}.$ 19. $\frac{3}{x-7}+\frac{7}{x+6}=\frac{x^2-36}{x^2-x-42}.$

20. $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{cd-ax}.$ 21. $\frac{x}{a}+\frac{a}{x}=\frac{a}{b}+\frac{b}{a}.$

22. $x+\frac{1}{x}=a+\frac{1}{a}.$

23. $x^2+2ax=b^2-a^2.$ 24. $x^2+a(1+3b)x+3a^2b=0.$

25. $(a-x)^2-(a-x)(b-x)+(x-b)^2=(a-b)^2$.

26. $a^2(x-b)^2=b^2(x-a)^2$.

27. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{c+a}+\dfrac{1}{c+b}$.

28. $\dfrac{1}{a}+\dfrac{1}{a+x}+\dfrac{1}{a+2x}=0$.

29. $(2a-b-x)^2-(a+b-2x)^2+9(a-b)^2=0$.

30. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{p}{q}$.

31. $\dfrac{(x-a)^2+(x-b)^2}{(a-x)(b-x)}=\dfrac{5}{2}$.

32. $\dfrac{x+a-b}{x-a+b}=\dfrac{a}{b}\left(\dfrac{x+a+5b}{x+5a+b}\right)$.

33. $\dfrac{x+a+2b}{x+a-2b}=\dfrac{b-2a+2x}{b+2a-2x}$.

34. $\dfrac{(x+a)^2+(x-b)^2}{(x+a)^2-(x-b)^2}=\dfrac{a^2+b^2}{2ab}$.

35. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{2c}{x-c}$.

36. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{1}{a}+\dfrac{1}{b}$.

37. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a^2+b^2}{ab}$.

38. $abx^2-(a^2+b^2)x+ab=0$.

39. $cx+\dfrac{ac}{a+b}=(a+b)x^2$.

40. $\dfrac{(x-1)}{(x+1)(x+2)}=\dfrac{(x+3)(x+4)}{(x+5)(x+6)}$.

41. $\dfrac{1}{x+a}+\dfrac{1}{x+2a}+\dfrac{1}{x+3a}=\dfrac{3}{x}$.

42. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

43. $(3a^2+b^2)(x^2-x+1)=(3b^2+a^2)(x^2+x+1)$.

44. $(ax-b)(bx-a)=c^2$.

45. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$.

46. $x+4\sqrt{x}=21$.

47. $x+\sqrt{5x+10}=8$.

48. $x-2\sqrt{(x^2+x+5)}-14=0$.

49. $2x+\sqrt{4x+8}=\dfrac{7}{2}$.

50. $\sqrt{2x+a}+\sqrt{x-a}=\sqrt{b}$.

51. $\sqrt{x+8}-\sqrt{x+3}=\sqrt{x}$.

52. $(4-2\sqrt{3})x^2-(\sqrt{3}-1)x=2$.

53. $x^2-(2+\sqrt{3})x+(7+4\sqrt{3})=0$.

54. $(27-10\sqrt{2})x^2+(5-\sqrt{2})x-2=0$.

55. $\sqrt{2x+9}+\sqrt{x-4}=\sqrt{6x+1}$.

56. $\sqrt{x+4}-\sqrt{2x-9}=\sqrt{x-1}$.

57. $\sqrt{3x+4}+\sqrt{2x+2}=\sqrt{10x+11}$.

58. $\sqrt{3x+3}-\sqrt{x+5}=\sqrt{x-7}$.

59. $\sqrt{x+1}+\sqrt{2x+3}=\sqrt{8x+1}$.

60. $\sqrt{3x+2}+\sqrt{6x+2}=\sqrt{12x+21}$.

61. $\dfrac{\sqrt{a+x}}{b}=\dfrac{a}{\sqrt{a+x}+b}$. 62. $\sqrt{5-x}+\sqrt{5+x}=\dfrac{12}{\sqrt{5+x}}$.

63. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{x+3}{x-3}$.

64. $\sqrt{(2x-1)}-\sqrt{(5x-4)}=\sqrt{(4x-3)}-\sqrt{(3x-2)}$.

65. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}$.

66. $\sqrt{x}-\sqrt{a}+\sqrt{(x+a-b)}=\sqrt{b}$.

67. $(x+a)(x+mb)(mx-b)=(mx+a)(x+b)(x-ma)$.

68. $x=\dfrac{3}{4-\dfrac{3}{4-\dfrac{3}{4-x}}}$. 69. $\dfrac{x-2}{x-4}-\dfrac{x-1}{x-3}=\dfrac{x-2}{x-1}-\dfrac{x-3}{x-2}$.

70. $1-x+x^2=\dfrac{1-b}{1+b}(1+x^2+x^4)$.

71. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}=\dfrac{4x-7}{2x-4}$.

72. $x(x-\sqrt{2}-\sqrt{3}-3)+\sqrt{2}+\sqrt{3}+2=0$.

73. $\dfrac{m^2-n^2}{m^2+n^2}=x+\dfrac{mn}{(m^2-n^2)x}$.

74. $\dfrac{a+m}{x-m}+m\dfrac{x+1}{x-1}+\dfrac{m+1}{m-1}=0$.

75. $\dfrac{1+a^2x^2}{1-ax}-\dfrac{1+b^2x^2}{1-bx}=(a-b)x.$ 76. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\dfrac{x}{a}.$

77. $\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}=ax.$

78. $\sqrt{(ax+b)}-\sqrt{(bx+a)}=\sqrt{(a+b)}.$

79. $\sqrt[3]{\{2x^2(x+1)+10x+1\}}=2x+1.$

80. $\dfrac{1-\sqrt{(x^2-1)}}{1+\sqrt{(x^2-1)}}=\dfrac{x-\sqrt{(x^2+8)}}{x+\sqrt{(x^2+8)}}.$

*154. Relations between the Roots and Co-efficients.

Let α and β be the roots of $ax^2+bx+c=0$; then

$$\alpha=\frac{1}{2a}(-b+\sqrt{b^2-4ac}) \text{ and } \beta=\frac{1}{2a}(-b-\sqrt{b^2-4ac}).$$

We have $\alpha+\beta=\dfrac{1}{2a}(-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac})$

$=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{1}{2a}(-b+\sqrt{b^2-4ac})\times\dfrac{1}{2a}(-b-\sqrt{b^2-4ac})$

$$=\frac{1}{4a^2}\{b^2-(b^2-4ac)\}=\frac{c}{a}.$$

Hence the following relations :—

i. *When the co-efficient of* x^2 *is unity, the sum of the roots = the co-efficient of the second term with the sign changed.*

ii. *The product of the roots is equal to the last term.*

***155.** *If* α *and* β *be the roots of* $x^2+px+q=0$, *then the expression* $x^2+px+q=(x-\alpha)(x-\beta)$.

Since $p=-(\alpha+\beta)$ and $q=\alpha\beta$.

$\therefore x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta=(x-\alpha)(x-\beta).$

Hence we can resolve any quadratic expression into two factors. We can also form the equation of which the roots are given.

We shall work out a few examples on the preceding articles.

Ex. 1. Find the sum and product of the roots of—

$$5x^2-7x+9=0.$$

If p and q be the roots, $p+q=\frac{7}{5}$ and $pq=\frac{9}{5}$...(Art. 154).

* May be omitted on first reading.

Ex. 2. Form the equation whose roots are -3 and 8.

The equation is $\{x-(-3)\}\ \{x-8\}=0$ or $x^2-5x-24=0$.

Ex. 3. Resolve into factors $12x^2-5x-3$.

The expression $=12(x^2-\frac{5}{12}x-\frac{1}{4})$.

The roots of the equation $x^2-\frac{5}{12}x-\frac{1}{4}=0$ are

$$\tfrac{1}{2}\left(\tfrac{5}{12}\pm\sqrt{\tfrac{25}{144}+1}\right)=\tfrac{1}{2}\left(\tfrac{5}{12}\pm\tfrac{13}{12}\right)=\tfrac{3}{4}\text{ or }-\tfrac{1}{3}.$$

$\therefore$ the factors of $x^2-\frac{5}{12}x-\frac{1}{4}$ are $(x-\frac{3}{4})$ and $(x+\frac{1}{3})$.

$\therefore$ the factors of $12(x^2-\frac{5}{12}x-\frac{1}{4})$ are $12(x-\frac{3}{4})(x+\frac{1}{3})$

$$=\frac{12(4x-3)(3x+1)}{4\times 3}=(4x-3)(3x+1).$$

Ex. 4. If α and β be the roots of $ax^2+bx+c=0$, find the value of $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$, in terms of a, b, c.

We know from Art. 154, $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2\frac{c}{a}=\frac{b^2}{a^2}-\frac{2c}{a}.$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\left(-\frac{b}{a}\right)^3-3\frac{c}{a}\left(-\frac{b}{a}\right)=$$

$$-\frac{b^3}{a^3}+\frac{3bc}{a^2}.$$

Ex. 5. Given *one* root of the equation $(b-c)x^2+(c-a)x+a-b=0$ to be 1, find the *other* root.

The product of the roots is $\frac{a-b}{b-c}$ and one root is 1.

$\therefore$ the other root is $\frac{a-b}{b-c}$.

EXERCISE 68.

Write down the *sum* and *product* of the roots of—

1. $x^2-5x+1=0$. 2. $3x^2+7x+3=0$. 3. $4x^2+3=0$.
4. $5x^2+7x=0$. 5. $(a+1)x^2+(a^2-1)x+a^3+1=0$.
6. $(a^2-b^2)x^2+(b^2-c^2)x+c^2-a^2=0$.

Form the *equation* whose roots are—

7. 4 and -3. 8. $-a$ and b. 9. $-a+1$ and $a+1$.

10. $a-b$ and $b-c$. 11. $-ab$ and $-bc$. 12. $2+\sqrt{3}$ and $2-\sqrt{3}$.

If α and β be the roots of $ax^2+bx+c=0$, find in terms of a, b, c the values of—

13. $\frac{1}{\alpha}+\frac{1}{\beta}$. 14. $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$. 15. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$. 16. $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$.

17. $\alpha^2-\alpha\beta+\beta^2$. 18. $\alpha^2+\alpha\beta+\beta^2$. 19. $\alpha^4+\alpha^2\beta^2+\beta^4$.

Resolve into *factors* : —[See Exercise 22 for more examples].

20. $(a^2-b^2)x^2+(b^2-c^2)x+c^2-a^2$.

21. $x^2+(a-c)x+(a-b)(b-c)$.

22. $(b+c-2a)x^2+(c+a-2b)x+a+b-2c$.

23. $a(b-c)x^2+b(c-a)x+c(a-b)$. 24. $x^2-8x+11$.

25. $x^2+(a+b)^2x+ab(a+b)^2$.

26. $x^2+(a-b)^2x-ab(a-b)^2$.

CHAPTER XXVII.

EQUATIONS SOLVED LIKE QUADRATICS.

156. I. Equations of the form $x^{2n} \pm p x^n = q$.

Ex. 1. Solve $x^4 - 6x^2 = 16$. $\therefore x^4 - 6x^2 - 16 = 0$.

Putting y for x^2, we have $y^2 - 6y - 16 = 0$.

$\therefore y = \frac{1}{2}(6 \pm \sqrt{36+64}) = \frac{1}{2}(6 \pm 10) = 8$ or -2.

$\therefore x^2 = 8$ or -2. $\therefore x = \pm\sqrt{8}$ or $\pm\sqrt{-2}$.

Ex. 2. Solve $x^3 - 4x^{\frac{3}{2}} = 21$. $\therefore x^3 - 4x^{\frac{3}{2}} - 21 = 0$.

Putting y for $x^{\frac{3}{2}}$, we have $y^2 - 4y - 21 = 0$.

$\therefore y = \frac{1}{2}(4 \pm \sqrt{16+84}) = \frac{1}{2}(4 \pm 10) = 7$ or -3.

$\therefore x^{\frac{3}{2}} = 7$ or -3. $\therefore x = (7)^{\frac{2}{3}}$ or $(-3)^{\frac{2}{3}}$.

Ex. 3. Solve $\sqrt{a+x} - 2\sqrt[4]{a+x} = 4$.

$\therefore \sqrt{a+x} - 2\sqrt[4]{a+x} - 4 = 0$.

Putting y for $\sqrt[4]{a+x}$, we have $y^2 - 2y - 4 = 0$.

$\therefore y = \frac{1}{2}(2 \pm \sqrt{4+16}) = 1 \pm \sqrt{5}$. $\therefore \sqrt[4]{a+x} = 1 \pm \sqrt{5}$.

$\therefore a + x = (1 \pm \sqrt{5})^4$, $x = (1 \pm \sqrt{5})^4 - a$.

Ex. 4. Solve $4^x + 2^{x+1} = 80$. $\therefore 2^{2x} + 2 \cdot 2^x - 80 = 0$.

Putting y for 2^x, we have $y^2 + 2y - 80 = 0$.

$\therefore (y+10)(y-8) = 0$. $\therefore y = 8$ or -10. $\therefore 2^x = 8$ or -10.

$\therefore 2^x = 2^3$. $\therefore x = 3$.

II. **Equations of the form** $X^{2n} \pm pX^n = q$, (in which X represents any simple or quadratic expression involving the unknown quantity).

Ex. 5. Solve $3x^2 + 15x - 2 = \sqrt{x^2 + 5x + 1}$.

$\therefore x^2 + 5x - \frac{2}{3} - \frac{1}{3}\sqrt{x^2 + 5x + 1} = 0$.

$\therefore x^2 + 5x + 1 - \frac{1}{3}\sqrt{x^2 + 5x + 1} - \frac{5}{3} = 0$.

Putting y for $\sqrt{x^2 + 5x + 1}$, we have $y^2 - \frac{1}{3}y - \frac{5}{3} = 0$.

$$\therefore y=\frac{1}{2}\left(\frac{1}{3}\pm\sqrt{\frac{1}{9}+\frac{20}{3}}\right)=\frac{1}{2}\left(\frac{1}{3}\pm\frac{\sqrt{61}}{3}\right)=\frac{1\pm\sqrt{61}}{6}$$

$=p$ (suppose).

$\therefore \sqrt{x^2+5x+1}=p.$ $\therefore x^2+5x+1=p^2.$

$\therefore x^2+5x+1-p^2=0.$ $\therefore x=\frac{1}{2}\{-5\pm\sqrt{25-4(1-p^2)}\}$
$=\frac{1}{2}\{-5\pm\sqrt{(21+4p^2)}\}.$

Ex. 6. Solve $(x^2+5x)^2-8x^2-40x-84=0.$

$\therefore (x^2+5x)^2-8(x^2+5x)-84=0.$

Putting y for x^2+5x, we have $y^2-8y-84=0.$

$\therefore (y-14)(y+6)=0.$ $\therefore y=14$ or $-6.$

$\therefore x^2+5x=14$ or $-6.$ $\therefore x^2+5x-14=0$ or $x^2+5x+6=0.$

$\therefore (x+7)(x-2)=0$ or $(x+2)(x+3)=0.$

$\therefore x=-7$ or 2 or -2 or $-3.$

III. **Equations of the form $(x+a)(x+b)(x+c)(x+d)=p$ when the sum of any two of the quantities a, b, c, d, is equal to the sum of the remaining two.**

Ex. 7. Solve $(x+1)(x+2)(x+3)(x+4)=24.$

Since $1+4=2+3$, we multiply together the first and last factors, and the second and third; hence

$$(x^2+5x+4)(x^2+5x+6)=24.$$

Putting y for x^2+5x+4, we have—

$y(y+2)=24.$ $\therefore y^2+2y-24=0.$ $\therefore (y+6)(y-4)=0.$

$\therefore y=4$ or $-6.$ $\therefore x^2+5x+4=4$ or $-6.$

$\therefore x^2+5x=0$ or $x^2+5x+10=0.$ $\therefore x=0$ or -5 or $\frac{1}{2}(-5\pm\sqrt{-15}).$

Ex. 8. Solve $(x+3)(x+5)(x-4)(x-6)=280.$

Since $3-4=5-6$, we multiply $(x+3)$ and $(x-4)$ together and $(x+5)$ and $(x-6)$ together; hence

$$(x^2-x-12)(x^2-x-30)=280.$$

Putting y for x^2-x-12, we have $y(y-18)-280=0.$

$\therefore y^2-18y-280=0.$ $\therefore (y-28)(y+10)=0.$

$\therefore y=28$ or $-10.$ $\therefore x^2-x-12=28$ or $-10.$

$\therefore x^2-x-40=0$ or $x^2-x-2=0.$

$\therefore x=\frac{1}{2}(1\pm\sqrt{161}),\ \frac{1}{2}(1\pm\sqrt{9})=\frac{1}{2}(1\pm\sqrt{161}),\ 2$ or $-1.$

IV. **Reciprocal Equations.** Every equation of the form $ax^4+bx^3+cx^2+bx+a=0$ in which the co-efficients of the terms equidistant from the beginning and end are equal is called a *reciprocal equation*, and may be solved as a quadratic by dividing both sides of the equation by x^2.

Ex. 9. Solve $x^4-3x^3+4x^2-3x+1=0$.

Dividing by x^2, we have $x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0$,

or $x^2+\frac{1}{x^2}+2-3\left(x+\frac{1}{x}\right)+2=0$

or $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+2=0$.

Putting y for $x+\frac{1}{x}$, we have—

$y^2-3y+2=0$. $\therefore (y-2)(y-1)=0$. $\therefore y=2$ or 1.

$\therefore x+\frac{1}{x}=2$ or 1. $\therefore x^2-2x+1=0$ or $x^2-x+1=0$.

$\therefore x=1, 1, \frac{1}{2}(1\pm\sqrt{-3})$.

Ex. 10. Solve $x^4+x^3-x^2-x+1=0$.

Dividing by x^2, we have $x^2+x-1-\frac{1}{x}+\frac{1}{x^2}=0$

or $x^2+\frac{1}{x^2}-2+x-\frac{1}{x}+1=0$.

or $\left(x-\frac{1}{x}\right)^2+\left(x-\frac{1}{x}\right)+1=0$.

Putting y for $x-\frac{1}{x}$, we have $y^2+y+1=0$.

$\therefore y=\frac{1}{2}(-1\pm\sqrt{-3})$. $\therefore x-\frac{1}{x}=\frac{1}{2}(-1\pm\sqrt{-3})=p$ (suppose).

$\therefore x^2-px-1=0$. $\therefore x=\frac{1}{2}(p\pm\sqrt{p^2+4})$.

Ex. 11. Solve $(x+1)^5=16(x^5+1)$.

Dividing by $x+1$, we have $(x+1)^4=16(x^4-x^3+x^2-x+1)$.
Reducing, $15x^4-20x^3+10x^2-20x+15=0$.

Dividing by $5x^2$, $3x^2-4x+2-\frac{4}{x}+\frac{3}{x^2}=0$

or $3\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+2=0$

or $3\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)-4=0$

Putting y for $x+\frac{1}{x}$, we have $3y^2-4y-4=0$

$\therefore\ y=\frac{1}{6}(4\pm 8)=2$ or $-\frac{2}{3}$. $\quad\therefore\ x+\frac{1}{x}=2$ or $-\frac{2}{3}$. Whence we find $x=1, 1, \frac{1}{3}(-1\pm 2\sqrt{-2})$, besides -1 which corresponds to the factor $(x+1)$ which we took out at first.

V. Miscellaneous Examples.

Ex. 12. Solve $\sqrt{x^2-5x+4}+\sqrt{x^2-4x+5}=x+1$......$(A)$

Identically, $(x^2-5x+4)-(x^2-4x+5)=-(x+1)$.........$(B)$

Dividing (B) by (A), $\sqrt{x^2-5x+4}-\sqrt{x^2-4x+5}=-1$.$(C)$

Adding (A) and (C), $2\sqrt{x^2-5x+4}=x$.

$\therefore\ 4(x^2-5x+4)=x^2$. $\quad\therefore\ 3x^2-20x+16=0$.

$\therefore\ x=\frac{1}{6}(20\pm 4\sqrt{13})=\frac{1}{3}(10\pm 2\sqrt{13})$.

Ex. 13. Solve $\sqrt{x^2-6x+15}+\sqrt{x^2-6x+13}$
$=\sqrt{(10)}-\sqrt{(8)}$..(A)

Identically, $(x^2-6x+15)-(x^2-6x+13)=2=10-8$......$(B)$

Dividing (B) by (A), $\sqrt{x^2-6x+15}-\sqrt{x^2-6x+13}$
$=\sqrt{(10)}+\sqrt{(8)}$..(C)

Adding (A) and (C), $2\sqrt{x^2-6x+15}=2\sqrt{(10)}$.

$\therefore\ x^2-6x+15=10$. $\quad\therefore\ x^2-6x+5=0$.

$\therefore\ (x-5)(x-1)=0$. $\quad\therefore\ x=5$ or 1.

Ex. 14. *To find the cube roots of unity.*

Let $x=\sqrt[3]{1}$; then $x^3=1$. $\quad\therefore\ x^3-1=0$.

or $(x-1)(x^2+x+1)=0$. $\therefore x-1=0$. $\therefore x=1$.

Also $x^2+x+1=0$. $\therefore x=\frac{1}{2}(-1\pm\sqrt{-3})$.

The cube roots are 1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

If $\frac{-1+\sqrt{-3}}{2}$ be denoted by w, then $w^2=\left(\frac{-1+\sqrt{-3}}{2}\right)^2$ $=\frac{1-3-2\sqrt{-3}}{4}=\frac{-1-\sqrt{-3}}{2}$. Hence *one imaginary root is the square of the other.* Also $1+w+w^2=1+\frac{-1+\sqrt{-3}}{2}$ $+\frac{-1-\sqrt{-3}}{2}=1-1=0$. *The sum of the roots* $=0$. *The product of the roots,* i.e., $1\times w\times w^2$ or $w^3=1\times\left(\frac{-1+\sqrt{-3}}{2}\right)$ $\times\left(\frac{-1-\sqrt{-3}}{2}\right)=\frac{1+3}{4}=1$.

Ex. 15. Solve $\sqrt{x-a}+\sqrt{x-b}+\sqrt{x-c}=0$.

If $p+q+r=0$, then $(p^2+q^2+r^2)^2=2(p^4+q^4+r^4)$.

Hence $(x-a+x-b+x-c)^2=2\{(x-a)^2+(x-b)^2+(x-c)^2\}$.

$\therefore 3x^2-2(a+b+c)x=a^2+b^2+c^2-2bc-2ac-2ab$.

$\therefore x=\frac{1}{3}\{a+b+c\pm2\sqrt{a^2+b^2+c^2-ab-ac-bc}\}$.

Ex. 16. Solve—

$$\frac{1}{6x^2-7x+2}+\frac{1}{12x^2-17x+6}=8x^2-6x+1,$$

or $\frac{1}{(2x-1)(3x-2)}+\frac{1}{(3x-2)(4x-3)}=(2x-1)(4x-1)$,

or $\frac{4x-3+2x-1}{(2x-1)(3x-2)(4x-3)}=(2x-1)(4x-1)$.

or $\frac{2}{(2x-1)(4x-3)}=(2x-1)(4x-1)$,

$\therefore (2x-1)^2(4x-1)(4x-3)=2$.

Putting y for $2x-1$, we have $y^2(2y+1)(2y-1)=2$.

$\therefore\ y^2(4y^2-1)=2. \qquad \therefore\ 4y^4-y^2-2=0.$

$\therefore\ y^2=\frac{1}{8}(1\pm\sqrt{33}). \qquad \therefore\ y=\pm\sqrt{\tfrac{1}{8}(1\pm\sqrt{33})}.$

$\therefore\ 2x-1=\pm\sqrt{\tfrac{1}{8}(1\pm\sqrt{33})}. \quad \therefore\ x=\tfrac{1}{2}\left\{1\pm\sqrt{\tfrac{1}{8}(1\pm\sqrt{33})}\right\}.$

EXERCISE 69.

Solve the following equations :—

1. $x^6-2x^3=48.$ 2. $x^4-2x^3+x=132.$
3. $2x^4-7x^2=99.$ 4. $x^{2n}-mx^n-p=0.$
5. $x+6\sqrt{x}=27.$ 6. $x-1=2+\dfrac{2}{\sqrt{x}}.$
7. $(x+m)^{\frac{2}{3}}-5(m^2-x^2)^{\frac{1}{3}}=-4(m-x)^{\frac{2}{3}}.$
8. $\sqrt[m]{(1+x)^2}-2\sqrt[m]{(1-x)^2}=\sqrt[m]{(1-x^2)}.$
9. $x^3+1=0.$
10. $(1+x^4)=a(1+x)^4.$ 11. $2x\sqrt{1-x^4}=a(1+x^4).$
12. $5^{2x}+6\times5^x=775.$ 13. $3^{4x}+2\times3^{2x}=783.$
14. $7^{2x}+2\times7^x=63.$ 15. $11^{4x}+21\times11^{2x}=143.$
16. $x^2+5x+3\sqrt{x^2+5x+7}+3=0.$
17. $x(x-3)+\sqrt{x^2+9x+3}=3(9-4x).$
18. $x^2+\sqrt{x^2-5}=11.$ 19. $x^2-8x-2\sqrt{x^2-8x+40}=-5.$
20. $x^2-x+3\sqrt{2x^2-3x+2}=\dfrac{x}{2}+7.$
21. $2x^2-\sqrt{x^2-2x-3}=4x+9.$
22. $x^2-7x+\sqrt{x^2-7x+18}=24.$
23. $x^2+3=2\sqrt{(x^2-2x+2)}+2x.$
24. $5\sqrt{x^2+5x+28}=x^2+5x+4.$
25. $(x+5)(x-2)+3\sqrt{x(x+3)}=0.$
26. $x(x+1)+3\sqrt{2x^2+6x+5}=25-2x.$
27. $\sqrt{(x^2-2x+9)}=3-x+\dfrac{x^2}{2}.$

28. $x^2+3-\sqrt{(2x^2-3x+2)}=\frac{3}{2}(x+1)$.

29. $x^2-x+3\sqrt{(2x^2-3x+2)}=7+\dfrac{x}{2}$.

30. $3x^2-2ax+5=2\sqrt{(3x^2-2ax+4)}$.

31. $x^{\frac{q+p}{2pq}}=\frac{1}{2}\dfrac{a^2-b^2}{a^2+b^2}\cdot\left(x^{\frac{1}{p}}+x^{\frac{1}{q}}\right)$.

32. $2^{2x}-3\times 2^{x+2}+32=0$.

33. $(9^x)^x-2(3^{x^2})(3^{x+1})=3^{2x+3}$.

34. $2^{2x+2}+4^{1-x}=17$.

35. $(1+x)^{\frac{2}{5}}+(1-x)^{\frac{2}{5}}=(1-x^2)^{\frac{1}{5}}$.

36. $x^4-10x^3+120x+144=0$.

37. $ax^4+bx^3+cx^2-bx+a=0$.

38. $2x^4-3x^3-4x^2+3x+2=0$.

39. $x^4-9x^3+81x+81=0$.

40. $x^4+x^3-4x^2+x+1=0$.

41. $x^4-2x^3+2x^2-2x+1=0$.

42. $x^4+3x+1=3x^3+\dfrac{4}{9}x^2$.

43. $x^2+\dfrac{1}{x^2}+2\left(x+\dfrac{1}{x}\right)=\dfrac{142}{9}$.

44. $6x^4-35x^3+6x^2-35x+6=0$.

45. $x^4-5x^3+5x^2-5x+4=0$.

46. $x^4+2x^3-11x^2+4x+4=0$.

47. $x^2+\dfrac{4}{x^2}-\dfrac{15x}{2}-\dfrac{15}{x}+\dfrac{35}{2}=0$.

48. $(x-1)(x-2)(x-3)(x-4)=120$.

49. $(x-2a)(x-3a)(x-4a)(x-5a)=360a^4$.

50. $16x(x-1)(x-2)(x-3)=9$.

51. $(x-a)(x-2a)(x-3a)(x-4a)=3a^4$.

52. $(x+a)(x+2a)(x+3a)(x+4a)=c^4$.

53. $x^4+ax^3+bx^2+cx+\dfrac{c^2}{a^2}=0$.

54. $(x+a)(x-a)(x-3a)(x-5a)=16c^4$.

55. $(x-3)(x-4)(x-5)(x-6)=24$.

56. $(x+3)(x+8)(x+13)(x+18)=51$.

57. $(x+a)(x+a+b)(x+a+2b)(x+a+3b)=c^4$.

58. $2x^2+2\sqrt{x^2+4x-5}=4x^2+8x+5.$

59. $(x+5)^{\frac{1}{2}}+(x+5)^{\frac{1}{4}}=2.$ 60. $\sqrt[4]{2}-\sqrt[4]{x-1}=\sqrt[4]{x+1}.$

61. $16x(x+1)(x+2)(x+3)=9.$

62. $(x-2)(x-3)(x-4)=6.$ 63. $(x-1)(x-2)(x-3)=24$

64. $\sqrt{2x+9}+\sqrt{3x-15}=\sqrt{7x+8}.$

65. $\sqrt{x^2+9}+\sqrt{x^2-9}=\sqrt{34}+4.$

66. $\sqrt{x^2+2x-1}+\sqrt{x^2+x+1}=\sqrt{2}+\sqrt{3}.$

67. $\sqrt{x^2+ax-1}+\sqrt{x^2+bx-1}=\sqrt{a}+\sqrt{b}.$

68. $x(x+2)(x+3)(x+5)=40.$

69. $(x^2+3x)^2-2x^2-6x-8=0.$

70. $2(4x^2-3x)^2-28x^2+21x+5=0.$

71. $(x^2-5x+7)^2-(x-2)(x-3)=1$

72. $(x^2+x)(x^2+x+1)=42.$

73. $\sqrt{3x^2+7x-5}-\sqrt{x^2+7x+3}=x-2.$

74. $\sqrt{x^2+3x+1}+\sqrt{x^2+4x-3}=2+\sqrt{x}.$

75. $3^{x+1}+9^x=108.$ 76. $x^4+2x^3+2x^2+x=\frac{1}{2}$

77. $x^4+\frac{97}{6}x^2+1-\frac{91}{12}x(x^2+1)=0.$

78. $(x+2)(x+4)(x+6)=(2x+1)(3x+2)(4x+3)$

79. $(x+4)(x+1)-5\sqrt{x(x+5)}=10.$

80. $\frac{1}{2x^2-5x+3}+\frac{1}{3x^2-5x+2}=6x^2-23x+21$

81. $\frac{1}{3x^2+11x+10}+\frac{1}{6x^2+19x+15}=\frac{2x^2+9x+10}{150}.$

82. $(x+3)^2-2(x^2+3)=2x(x+1)^2$

83. $x^{\frac{5}{12}}-\frac{1}{2}\frac{p^2-q^2}{p^2+q^2}(\sqrt{x}+\sqrt[3]{x})=0.$

84. $(x+m)^{\frac{2}{5}}+(x-m)^{\frac{2}{5}}=\left(n+\frac{1}{n}\right)(x^2-m^2)^{\frac{1}{5}}$

85. $x^2+x+10\sqrt{(x^2+3x+16)}=2(20-x).$

86. $x^3+x\sqrt{x}=72.$ 87. $9x-3x^2+4\sqrt{(x^2-3x+5)}=0.$

88. $x^3+6x^2-12x+8=0.$ 89. $\frac{b^2-x^2}{b^2-m^2}=\frac{x}{m}\cdot\frac{a-m}{a-x}.$

90. $\left(x-\dfrac{a^2}{x}\right)\left(x-\dfrac{b^2}{x}\right)\left(x-\dfrac{c^2}{x}\right)$
$$=(x-a)(x-b)(x-c).$$

91. $\dfrac{x+b}{a}+\dfrac{x+a}{b}=\dfrac{a}{x+b}+\dfrac{b}{x+a}.$

92. $\left(\dfrac{x^2-13x+31}{x^2-x-11}\right)^2+3\dfrac{x-3}{x+1}=0.$

93. $(x-4)^2-7=2\left(\dfrac{1}{x}-x\right).$

94. $\dfrac{x+a+\sqrt{x^2+a^2}}{\sqrt{x}+\sqrt{x^2+a^2}}=2\sqrt{x}.$ 95. $x^4-6x^2+8x-3=0.$

96. $81x(x+1)(x+2)(x+3)=40.$

97. $\dfrac{1}{15x^2-11x+2}+\dfrac{1}{10x^2-9x+2}=18x^2-9x+1.$

98. $\dfrac{1}{4x^2-9x+5}+\dfrac{1}{4x^2-7x+3}=8(x-1)^2.$

99. $\dfrac{4}{x^2-4x+3}-\dfrac{2}{x^2-3x+2}=x^2-3x+2.$

100. $\dfrac{(x+1)(x+2)}{(x-1)(x-2)}+\dfrac{(x-1)(x-2)}{(x+1)(x+2)}=\dfrac{(x+3)(x+4)}{(x-3)(x-4)}$
$$+\frac{(x-3)(x-4)}{(x+3)(x+4)}.$$

CHAPTER XXVIII.

SIMULTANEOUS QUADRATIC EQUATIONS.

157. Simultaneous Equations: where one or more may be of a degree higher than the first.

We shall work out some examples.

Ex. 1. Solve $x^2+y^2=25$, $xy=12$.

From these $(x+y)^2=49$ and $(x-y)^2=1$.

$\therefore\ x+y=\pm 7$ and $x-y=\pm 1$.

Hence $\left.\begin{array}{l} x+y=7 \\ x-y=1 \end{array}\right\}$, $\left.\begin{array}{l} x+y=-7 \\ x-y=1 \end{array}\right\}$, $\left.\begin{array}{l} x+y=7 \\ x-y=-1 \end{array}\right\}$, $\left.\begin{array}{l} x+y=-7 \\ x-y=-1 \end{array}\right\}$.

$\therefore\ x=4, -3, 3, -4$ and $y=3, -4, 4, -3$.

Ex. 2. Solve $x^4+y^4=97$, $xy=6$.

$x^4+y^4=(x^2+y^2)^2-2x^2y^2$. $\quad\therefore\ (x^2+y^2)^2-2\times 36=97$.

$\therefore\ (x^2+y^2)^2=25 \quad \therefore\ x^2+y^2=\pm 5$ and $xy=6$.

From these the values of x and y can be found as in the last example.

Ex. 3. Solve $x^5+y^5=33$, $x+y=3$.

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4=\frac{33}{3}=11$$

or $(x^4+y^4+2x^2y^2)-xy(x^2+y^2)-x^2y^2=11$

or $(x^2+y^2)^2-xy(x^2+y^2)-x^2y^2=11$,

but $x^2+y^2=9-2xy$, $\quad\because\ x+y=3$.

$\therefore\ (9-2xy)^2-xy(9-2xy)-x^2y^2=11$

or $5x^2y^2-45xy+70=0$ or $x^2y^2-9xy+14=0$

or $(xy-7)(xy-2)=0$. $\quad\therefore\ xy=7$ or 2 and $x+y=3$.

Hence x and y can be found.

Note.—The following *identities* will help us in solving similar equations. The student will have no difficulty in verifying them.

$$\left.\begin{array}{ll} \text{i.} & x^2+y^2=p^2-2q \\ \text{ii.} & x^3+y^3=p^3-3pq \\ \text{iii.} & x^4+y^4=p^4-4p^2q+2q^2 \\ \text{iv.} & x^5+y^5=p^5-5p^3q+5pq^2 \end{array}\right\}\ \text{if } x+y=p \text{ and } xy=q.$$

$$\left.\begin{array}{ll}\text{i.} & x^2+y^2=p^2+2q \\ \text{ii.} & x^3-y^3=p^3+3pq \\ \text{iii.} & x^4+y^4=p^4+4p^2q+2pq^2 \\ \text{iv.} & x^5-y^5=p^5+5p^3q+5pq^2\end{array}\right\}\text{ if } x-y=p \text{ and } xy=q.$$

Ex. 4. Solve $x^5-y^5=211$ and $x-y=1$.

We know $x^5-y^5=p^5+5p^3q+5pq^2$ if $x-y=p$ and $xy=q$.

Since $x-y=1$, $x^5-y^5=1+5q+5q^2$.

$\therefore\ 5q^2+5q+1=211$ or $5q^2+5q-210=0$

or $q^2+q-42=0$ or $(q+7)(q-6)=0$.

$\therefore\ q=-7$ or 6, i.e., $xy=-7$ or 6 and $x-y=1$.

Hence x and y can be found.

Ex. 5. Solve $2x-y=3$............(1).

$3x^2-6xy+y^2=1$.....(2).

From (1), $y=2x-3$. Substituting in (2),

$3x^2-6x(2x-3)+(2x-3)^2=1$. $\quad\therefore\ 5x^2-6x-8=0$.

$\therefore\ x=\frac{1}{10}(6\pm14)=2$ or $-\frac{4}{5}$. $\quad$ Hence $y=1$ or $-\frac{23}{5}$.

158. Homogeneous Equations: when the terms containing x and y are of the same degree in each.

Ex. 6. Solve $x^2+xy=21$, $y^2-xy=4$.

Let $y=vx$; then these equations become—

$x^2(1+v)=21$........(1). $\quad x^2(v^2-v)=4$........(2).

By division, $\frac{1+v}{v^2-v}=\frac{21}{4}$. $\quad\therefore\ 4+4v=21v^2-21v$.

$\therefore\ 21v^2-25v-4=0$. $\quad\therefore\ v=\frac{1}{42}(25\pm31)=\frac{4}{3}$ or $-\frac{1}{7}$.

Hence from (1), $x^2\left(1+\frac{4}{3}\right)=21$, or $x^2\left(1-\frac{1}{7}\right)=21$.

$\therefore\ x=\pm3$, or $x=\pm\frac{7}{\sqrt{2}}$. Since $y=vx$, $\therefore\ y=\pm4$ or $\mp\frac{1}{\sqrt{2}}$.

Ex. 7. Solve $5xy-2x^2=12$; $3y^2+xy=18$.

Let $y=vx$; then these equations become—

$x^2(5v-2)=12$........(1). $\quad x^2(3v^2+v)=18$........(2).

By division, $\frac{3v^2+v}{5v-2}=\frac{3}{2}$ $\quad\therefore\ 6v^2+2v=15v-6$.

$\therefore\ 6v^2-13v+6=0.\qquad \therefore\ v=\frac{1}{12}(13\pm5)=\frac{3}{2}\text{ or }\frac{2}{3}.$

Hence from (1), $x^2\left(\frac{15}{2}-2\right)=12$, or $x^2\left(\frac{10}{3}-2\right)=12.$

$\therefore\ x=\pm\sqrt{\frac{24}{11}}$ or $x=\pm3$. Since $y=vx$,

$$\therefore\ y=\pm\sqrt{\frac{54}{11}}\text{ or }\pm2.$$

EXERCISE 70.

Solve the following equations:—

1. $y+\frac{\sqrt{y}}{x}=\frac{42}{x};\ \frac{x^2}{3}+\frac{x}{2\sqrt{y}}=\frac{54}{y}.$
2. $(x^2+y^2)xy=30;\ x^4+y^4=82.$
3. $x^2+y^2=10;\ (x+1)(y+1)=8.$
4. $x^4+x^2y^2+y^4=21;\ x+y=7.$
5. $x^2+6xy+17y^2=33;\ 3xy+16y^2=22.$
6. $xy-y^2+2x-y=2;\ x-y^3=1.$
7. $x^2+xy-8y=1;\ x-y=-3.$
8. $\frac{1}{x+1}+\frac{1}{1-y}=\frac{7}{12};\ x-y=xy-13.$
9. $x(9-xy)=y(xy-36);\ xy(3x+12y-xy)=108(x+y-3).$
10. $4y^2+3xy-x^2=0;\ (y-a)(x-y)=(x+2a)(x+y).$
11. $x^2-2xy=7;\ 2y^2-xy=-3.$
12. $x^2+y^2+x+y)=26;\ 4(x+y=3xy.$
13. $6x+5y=\frac{6}{x}+\frac{5}{y}+29\frac{1}{3};\quad 3x+4y=\frac{3}{x}+\frac{4}{y}+18\frac{2}{3}.$
14. $y+\sqrt{x^2-1}=2;\ \sqrt{x+1}+\sqrt{x-1}=\frac{2}{\sqrt{y}}.$
15. $x+\frac{3}{y}=2;\ y+\frac{3}{x}=-2.$ 16. $\left.\begin{array}{r}x^2+y^2=2a^2;\\ x+y:x-y=m:n\end{array}\right\}.$
17. $x^2+y^2=a^2;\ x+y=b.$
18. $x-y=1;\ (x^2+xy+y^2)(x^3-y^3)=361.$
19. $2x+y=4x^2;\ 3y-2x=y^2.$
20. $xy+y^2=3;\ x^4-y^4=6(x^2+3y^2)-27.$

21. $x+y=a$; $x^5+y^5=b$.

22. $(1+m^2)(x+y)=2m(1+xy)$; $(1+n^2)(x-y)=2n(1-xy)$.

23. $49(x^2+a^2)(y^2+b^2)=50(xy+ab)^2$; $(x^2-a^2)(y^2-b^2)$
$=24(bx-ay)^2$. 24. $x^5-y^5=31$; $x-y=1$.

25. $4(x^2+y^2)-5xy+3(x+y)-37=0$; $3(x^2+y^2)-2xy$
$+4(x+y)-47=0$. 26. $3x-y=5$; $2x^2-4xy+y^2=1$.

27. $3x+2y=5$; $4x^2+9y^2=13$.

28. $x^2+xy=126$; $y^2+xy=198$. 29. $x^3-y^3=63$; $x-y=3$.

30. $x^2+xy+y^2=84$; $x-\sqrt{xy}+y=6$.

31. $x^4+x^2y^2+y^4=243$; $x^2+xy+y^2=9$.

32. $\sqrt[3]{x}+\sqrt[3]{y}=6$; $x+y=126$.

33. $(x^2+y^2)(x^3+y^3)=45$; $x+y=3$.

34. $x^2+2y^2=17$; $3x^2-5xy+4y^2=13$.

35. $x^3+xy^2=a^3$; $y^3+yx^2=b^3$.

36. $\dfrac{x^2}{a}+\dfrac{y^2}{b}=\dfrac{a^2}{x}+\dfrac{b^2}{y}=a+b$. 37. $\dfrac{x^2}{y}+\dfrac{y^2}{x}=9$; $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{4}$.

38. $x^2-xy+y^2=21$; $y^2-2xy+15=0$.

39. $x^4+y^4=14x^2y^2$; $x+y=a$.

40. $\dfrac{a}{a+x}+\dfrac{b}{b+y}=1$; $x+y=a+b$.

159. We add a few examples of quadratic equations involving *three* unknown quantities.

Ex. 1. Solve $x+y+z=x^2+y^2+z^2=x^3+y^2+z^3=1$.

Since $x+y+z=1$. $\therefore x^2+y^2+z^2+2(xy+yz+zx)=1$.

and $x^2+y^2+z^2 \qquad =1$.

$\therefore 2\,(xy+yz+zx)=0$. $\therefore$ $xy+yz+xz=0$..............(1).

Again, since $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)$. $\therefore$ $1-3xyz=1\times(1-0)=1$. $\therefore 3xyz=0$. $\therefore$ $xyz=0$...(2)

Now $(1-x)(1-y)(1-z)=1-(x+y+z)+(xy+yz+zx)-xyz$ $=1-1+0-0=0$.

$\therefore$ one of the factors $1-x$, $1-y$, $1-z$ must $=0$.

Let $1-x=0$; then $x=1$; $\therefore$ $y+z=0$. $\because$ $x+y+z=1$

and from (2), $yz=0$. $\therefore$ $y=0$ and $z=0$.

Ex. 2. Solve $x^2+2yz=13$......(i), $y^2+2xz=10$........(ii) and $z^2+2xy=13$......(iii).

Adding, $x^2+y^2+z^2+2xy+2yz+2zx=36$.

$\therefore\ x+y+z=6$..........................(iv).

Subtracting (iii) from (i), $x^2-z^2+2y(z-x)=0$.

$\therefore\ (x-z)(x+z-2y)=0$. $\therefore\ x+z=2y$. $\therefore$ from (iv), $3y=6$.

$\therefore\ y=2$. $\therefore\ x+z=4$ and from (ii), $xz=\frac{1}{2}(10-4)=3$.

$\therefore\ x=3$ or 1 and $z=1$ or 3.

Ex. 3. Solve $x+y+z=6$; $x^3+y^3+z^3=36$ and $(x+z)\times(y+z)=20$.

Since $x+y+z=6$, $\therefore\ x^3+y^3+z^3+3(x+y)(y+z)(z+x)=216$.

But $x^3+y^3+z^3=36$ and $(x+z)(y+z)=20$.

$\therefore\ 36+3\times 20(x+y)=216$. $\therefore\ x+y=3$.

$\therefore\ z=3$ $\because\ x+y+z=6$.

From the last equation, $(x+3)(y+3)=20$.

$\therefore\ xy+3(x+y)=11$. $\therefore\ xy=11-9=2$ and $x+y=3$.

$\therefore\ x=2$ or 1 and $y=1$ or 2.

Ex. **4.** Solve $x+y+z=6$...(i); $x^2+y^2+z^2=14$...(ii) and $yz-x^2=5$..........................(iii).

From (ii) and (iii), $x^2+y^2+z^2+2(yz-x^2)=14+10$.

$\therefore\ (y+z)^2-x^2=24$. $\therefore\ (y+z+x)(y+z-x)=24$; but $x+y+z=6$, $\therefore\ y+z-x=4$ and $x+y+z=6$

$\therefore\ x=1$ and $y+z=5$,

and from (iii), $yz=6$. $\therefore\ y=3$ and $z=2$.

Ex. 5. Solve $x^2-yz=a$....(i); $y^2-xz=b$...(ii) and $z^2-xy=c$......(iii).

$(x^2-yz)^2-(y^2-xz)(z^2-xy)=a^2-bc$.

$\therefore\ x(x^3+y^3+z^3-3xyz)=a^2-bc$.

Similarly $y(x^3+y^3+z^3-3xyz)=b^2-ac$ and

$z(x^3+y^3+z^3-3xyz)=c^2-ab$

$$\therefore\ \frac{x}{a^2-bc}=\frac{y}{b^2-ac}=\frac{z}{c^2-ab}.$$

$$\therefore\ \text{each}=\frac{\sqrt{x^2-yz}}{\sqrt{(a^2-bc)^2-(b^2-ac)(c^2-ab)}}$$

$$=\pm\frac{\sqrt{a}}{\sqrt{a(a^3+b^3+c^3-3abc)}}$$

19. $\pm\dfrac{1}{\sqrt{(a^3+b^3+c^3-3abc)}}$ $\therefore\ x=\pm\dfrac{a^2-bc}{\sqrt{(a^3+b^3+c^3-3abc)}}$.

20.

Ex. 6. Solve $x^2+xy+y^2=19$...(i); $y^2+yz+z^2=37$...(ii) and $z^2+zx+x^2=28$......(iii).

From (i) and (ii), $x^2-z^2+y(x-z)=-18$.

$\therefore\ (x-z)(x+y+z)=-18$..........(iv).

From (i) and (iii), $y^2-z^2+x(y-z)=-9$.

$\therefore\ (y-z)(x+y+z)=-9$............(v).

By division, $\dfrac{x-z}{y-z}=2$. $\therefore\ x-z=2y-2z$. $\therefore\ y=\dfrac{x+z}{2}$.

Substituting in (ii), $\dfrac{(x+z)^2}{4}+\dfrac{z}{2}(x+z)+z^2=37$.

$\therefore\ 7z^2+4xz+x^2=148$,

and $z^2+zx+x^2=28$. Let $z=vx$; these equations become $x^2(7v^2+4v+1)=148$...(A) and $x^2(v^2+v+1)=28$...(B)

By division, $\dfrac{7v^2+4v+1}{v^2+v+1}=\dfrac{148}{28}=\dfrac{37}{7}$. $\therefore\ 12v^2-9v-30=0$.

$\therefore\ 4v^2-3v-10=0$. $\therefore\ (v-2)(4v+5)=0$. $\therefore\ v=2$ or $-\frac{5}{4}$.

Hence from (A), $x^2(28+8+1)=148$. $\therefore\ x=\pm2$. $\therefore\ z=\pm4$ and $y=\pm\dfrac{4+2}{2}=\pm3$. From the other value of v we can get two more values for each of x, y, z.

EXERCISE 71.

Solve the following equations :—

1. $x^2+y^2+z^2=6$; $xy+yz+zx=5$ and $xyz=2$.

2. $\dfrac{a^2}{x}=\dfrac{b^2}{y}=\dfrac{c^2}{z}=x+y+z$.

3. $x+y+z=7$; $x(y+z)=12$ and $x^2+y^2+z^2=21$.

4. $x^2+y^2+z^2=1$; $ax^2+by^2+cz^2=0$ and $bcx^2+acy^2+abz^2=0$.

5. $x+y+z=0$; $(b+c)x+(c+a)y+(a+b)z=0$; $x^3+y^3+z^3=3(a-b)(b-c)(c-a)$.

6. $x+y+z=6$; $x^2+y^2+z^2=14$; $x^3+y^3+z^3=36$.

7. $x+y+z=9$; $x^2+y^2+z^2=29$; $x^3+y^3+z^3=99$.

8. $x^3+y^3+z^3=36$; $(x+y)(y+z)(z+x)=60$; $(x+y+z)\times(xy+yz+zx)=66$.

9. $x^2+xy+y^2=c$; $y^2+yz+z^2=a$; $z^2+zx+x^2=b$.

10. $(y-a)(z-a)=bc$; $(z-b)(x-b)=ca$; $(x-c)(y-c)=ab$.

11. $x+y+z=0$, $ax+by+cz=0$, $x^3+y^3+z^3=3(a-b)(b-c)$
$\times(c-a)$.

12. $x(y+z-x)=a$; $y(x+z-y)=b$, $z(x+y-z)=c$.

13. $x^{-2}y^{-1}z=a$, $x^{-1}yz^2=b$, $x^2y^2z^3=c$.

14. $x^2+(y-z)^2=a$, $y^2+(z-x)^2=b$, $z^2+(x-y)^2=c$.

15. $x^2+yz=y^2+zx=z^2+xy=c$.

16. $x+y+z=a+b+c$, $x^2+y^2+z^2=a^2+b^2+c^2$,
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3.$$

17. $\dfrac{y+z}{a}=\dfrac{z+x}{b}=\dfrac{x+y}{c}=\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}$.

18. $x^2+2yz=148$, $y^2+2xz=145$, $z^2+2xy=148$.

19. $x^2y^2+y^2z^2+z^2x^2=49$, $x^2+y^2+z^2=14$; $x(y+z)=9$.

20. $x+y+z=1$, $x^2+y^2+z^2=13$, $x^3+y^3+z^3=19$.

CHAPTER XXIX.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

160. We shall solve some problems which lead to quadratic equations.

Ex. 1. Divide 15 into two parts, such that their product shall be 56.

Let x = one part; then $15-x$ = the other part.

By the condition of the problem, $x(15-x)=56$,

or $x^2-15x+56=0$. $\therefore (x-7)(x-8)=0$.

$\therefore x=7$ or 8 = *one* part; and 8 or 7 = the *other* part.

Ex. 2. Divide a given line into two parts such that the square on one part may be equal to the rectangle contained by the whole and the other part.

Let a denote the line, and x the length of one part; then $a-x$ is the length of the other part. By the question, $x^2=a(a-x)$

or $x^2+ax-a^2=0$. $\therefore x=\frac{1}{2}(-a\pm\sqrt{5a^2})=\frac{a}{2}(-1\pm\sqrt{5})$.

$$\therefore \frac{a}{2}(-1+\sqrt{5}) \text{ is the length of one part.}$$

The *negative* answer is the solution of the following problem: *produce a given line, so that the square on the given line may be equal to the rectangle contained by the whole line produced and the part produced.*

Ex. 3. A man buys a certain number of oxen for £100; if he had bought 5 more for the same money, each would have cost him £1 less; how many did he buy?

Let x = number of oxen; then $\frac{100}{x}$ = price of each ox in *pounds*.

If he had bought 5 more, $\frac{100}{5+x}$ would have been the price of each. By the question, $\frac{100}{5+x}=\frac{100}{x}-1$ or $100\left(\frac{1}{x}-\frac{1}{5+x}\right)=1$

or $x^2+5x-500=0$. $\therefore x=20$ or -25. $\therefore$ the number of oxen $=20$.

The latter root must be rejected.

Ex. 4. The length of a rectangular field exceeds its breadth by 33 yards. Its area is one acre; find the dimensions of its sides.

Let x = the breadth in *yards*; then $x+33$ = the length in *yards*. By the question, $x(x+33)=4840$ (one acre equals 4840 yards) or $x^2+33x-4840=0$. $\therefore x=-88$ or 55.

The negative value must be rejected.

The breadth $=55$ yards and the length $=88$ yards.

Ex. 5. A and B started at the same time for a place 300 miles distant. A travels a mile an hour faster than B, and arrives at his journey's end 10 hours before him; find the rate per hour at which each person travelled.

Let x = *number of miles per hour* that B travels;

then $x+1$ = *number of miles per hour* that A travels.

Now $\frac{300}{x}$ and $\frac{300}{x+1}$ denote the times in hours taken by B and A respectively. By the question, $\frac{300}{x+1}=\frac{300}{x}-10$

or $30\left(\frac{1}{x}-\frac{1}{x+1}\right)=1$. $\therefore x^2+x-30=0$. $\therefore (x+6)(x-5)=0$.

$\therefore x=5$ or -6. The negative value must be rejected.

$\therefore$ B's rate is 5 miles per hour; A's rate is 6 miles per hour

Ex. 6. A person sells a horse for £24 and gains as much per cent. as the horse cost him: what did the horse cost him?

Let x = the cost price of the horse in *pounds*; then his gain $=x\times\frac{x}{100}=\frac{x^2}{100}$.

By the question, $x+\frac{x^2}{100}=24$. $\therefore x^2+100x-2400=0$,

or $(x+120)(x-20)=0$. $\therefore x=-120$ or 20. The negative value should be rejected. The cost price of the horse is £20.

EXERCISE 72.

1. The difference of two numbers is 4, the difference of their squares is 40; find the numbers.

2. Find two *consecutive* numbers whose product is 420.

3. The difference of the cubes of two *consecutive* numbers is 217; find them.

4. Find two numbers whose sum, product and difference of squares, shall be equal.

5. Find two numbers each of which is the square of the other.

6. If three feet be taken from one side of a rectangle and added to the other side, its area is doubled; the sum of the sides is 7 feet; find the sides.

7. There is a rectangular field of 1 acre, whose length exceeds its breadth by 66 yards; find its dimensions.

8. There is a rectangular field whose length exceeds its breadth by 16 yards and it contains 960 square yards; find its dimensions.

9. The area of a rectangular field is $7\frac{1}{2}$ acres and the sum of the lengths of the two adjacent sides exceeds the length of either diagonal by 110 yards; find the length of the sides.

10. By selling a horse for £25 I lose as much per cent. as it cost me; what was its prime cost?

11. A person bought a certain number of oxen for £240 and after losing 3 sold the rest for £8 a head more than they cost him, thus gaining £59 by the bargain; how many did he buy?

12. A person bought 2 flocks of sheep for £15, in one of which there were 5 more than in the other; each sheep in each flock cost as many shillings as there were sheep in the other flock; how many were there in each?

13. *A* and *B* distribute £5 each in charity; *A* relieves 5 persons more than *B*, and *B* gives to each 1 shilling more than *A*; how many did each relieve?

14. Find three numbers, such that if the first be multiplied by the sum of the second and the third, the second by the sum of the first and the third, and the third by the sum of the first and the second, the products shall be 26, 50 and 56.

15. I have to walk a distance of 144 miles, and I find that if I increase my speed $1\frac{1}{2}$ miles per hour, I can do the journey in 16 hours less than if I walk at my usual rate; find my usual rate of walking.

16. There are two square fields, the greater of which is 4 times the less, and the side of the greater is 20 yards longer than the side of the other; find the area of each field.

17. The sum of the reciprocals of two numbers is $\frac{7}{12}$ and the product of the numbers is 12; find them.

18. A number of soldiers are formed into a hollow square of 3 deep; if 324 men are added to the centre they will form a solid square; find the whole number of men.

19 If a number consisting of two digits be divided by the sum of the digits, the quotient is the first digit, and the remainder the last; the product of the digits is 21; find the number.

20. The sum of a fraction and its reciprocal is $2\frac{1}{2}{}^{6}_{1}$; but if the numerator and the denominator be each increased by unity, the sum would then be $2\frac{1}{2}$; find the fraction.

21. A boat's crew can row both up and down a stream 16 miles long, flowing at the rate of 3 miles an hour, in 10 hours; find the rate of the crew in still water.

22 A number consists of three digits whose sum is 14; the square of the middle digit is equal to the product of the extreme digits, and if 594 be added to the number, the digits are *reversed*; find the number.

23. The united ages of a father and son amount to 64. Twice the father's age exceeds the square of the son's age by 8; find their ages.

24. Two kinds of oranges are sold in the market, two more of one kind being given for a shilling than of the other a score of the inferior sort costs $4d$ more than a dozen of the superior sort; find the price of the oranges.

25. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

26. The floor of a room contains 40 square yards, its height is 5 yards, and the length is 3 yards more than the breadth find the area of the 4 walls.

27. A man buys a horse which he sells again for £56 and gains as many pounds in £100 as the horse cost him; how much did he give for the horse?

28. A and B started at the same time for a place distant 150 miles. A travels 3 miles an hour faster than B and arrives at his journey's end 8 hours 20 minutes before B; find their rates.

29. A person bought oxen for £33-15-0 which he sold again at £2-8-0 a head, gaining thereby as much as one ox cost him; how many oxen did he buy?

30. Divide the number 30 into 2 parts such that their product may be equal to 125

31. *A* and *B* set out from two towns 247 miles distant from each other. *A* travelled at the rate of 9 miles a day, and the number of days at the end of which they met was greater by 3 than the number of miles which *B* went in a day; find the number of miles each travelled.

32. The fore-wheel of a carriage makes 64 revolutions more than the hind-wheel in passing over 1 mile: but if the circumference of the fore-wheel be increased by 11 inches, it will make only 40 revolutions more than the hind-wheel in the same space; find the circumference of each wheel.

33. One man can reap a field in 5 days less than another, and if they work together they can do it in 6 days, find in what time each could do it alone.

34. A man has to travel a certain distance; when he has gone 20 miles he increases his speed 1 mile per hour: if he had travelled at this increased rate during the whole of his journey he would have arrived 40 minutes earlier; but if he had kept on at his first rate he would have arrived 20 minutes later: find the length of the journey.

35. *A* and *B* engaged to reap equal quantities of wheat, and *A* began half an hour before *B*; they stopped at 12 o'clock and rested an hour, observing that half the work was done; *B*'s part was finished at 7 o'clock and *A*'s at a quarter before 10. Supposing them to have labored uniformly, find when they began.

36. A cask P is filled with 50 gallons of water, and a cask Q with 40 gallons of wine; X gallons are drawn from each cask, mixed and replaced. The same operation is repeated; find X when there are $8\frac{7}{8}$ gallons of wine in P after the second operation.

37. Divide a given line into two parts such that twice the square on one part may be equal to the rectangle contained by the whole line and the other part.

38. Find that number whose square added to its cube is 9 times the next higher number.

39. A farmer wishes to enclose a rectangular piece of land containing 1 acre 32 perches with 176 hurdles, each two yards long; how many hurdles must he place in each side of the rectangle?

40. What are eggs a dozen when two more in a shilling's worth lowers the price one penny per dozen?

41. A person rents a certain number of acres of land for £84; he cultivates 4 acres himself, and letting the rest for 10*s.* an acre more than he pays for it, receives for this portion the whole rent £84; find the number of acres.

42. *A* set off from London to York, and *B* at the same time from York to London, and they travel uniformly; *A* reaches York 16 hours, and *B* reaches London 36 hours, after they have met on the road; find in what time each has performed the journey.

43. A vessel can be filled with water by two pipes; by one of these pipes alone the vessel would be filled 2 hours sooner than by the other; also the vessel can be filled by both pipes together in $1\frac{7}{8}$ hours; find the time which each pipe would take to fill the vessel.

44. A person buys a quantity of wheat which he sells so as to gain 5 per cent. on his outlay, and thus clears £16. If he had sold it at a gain of 5 shillings per quarter, he would have cleared as many pounds as each quarter cost him shillings; find how many quarters he bought, and what each quarter cost him.

45. Two workmen, *A* and *B*, were employed by the day at different rates. *A* at the end of a certain number of days received £4-16-0, but *B*, who was absent six of those days, received only £2-14-0. If *B* had worked the whole time, and *A* had been absent six days, they would have received exactly alike; find the number of days, and what each was paid per day.

46. *A* and *B* run a race round a two-mile course. In the first heat *B* reaches the winning post 2 minutes before *A*. In the second heat *A* increases his speed 2 miles per hour, and *B* diminishes his as much; and *A* then arrives at the winning post 2 minutes before *B*; find at what rate each man ran in the first heat.

47. A person bought two pieces of cloth of different sorts; the finer cost 4 shillings a yard more than the coarser and he bought 10 yards more of the coarser than of the finer. For the finer piece he paid £18, and for the coarser piece £16; find the number of yards in each piece.

48. A merchant sent to his agent a certain number of bags of coffee for sale, expecting them to realize £250, and after the agent's commission of 4 per cent. was paid, to yield a profit of 20 per cent. on the original cost. On the way 5 bags were lost,

but the rest were sold at 10*s*. per bag above the estimated price. This raised the net profits £76-17-6 per cent. although the agent now received a commission of 5 per cent.; how many bags were there at first?

49. The relative value of two sorts of mixed metals consisting of gold and silver is as 11 to 17. If the proportion of gold to silver in each had been doubled, their relative value would have been as 7 to 11. The value of gold to that of silver being as 13 to 1; find the proportion of gold to silver in each of the mixed metals.

50. A person bought one horse for x£ and another for y£, he sold the first at a profit of x per cent. and the second at a loss of y per cent., and thus received $\frac{1}{5}$ as much again as he would have, had he sold the *first* at a profit of y per cent. and the *second* at a loss of x per cent. If he had bought x horses at x£ each, and sold them at x per cent. profit and had bought y horses at y£ each, and had sold them at y per cent. loss, he would have gained altogether £1,520; what did he give for each horse?

CHAPTER XXX.

MISCELLANEOUS THEOREMS AND EXAMPLES.

161. Method of Detached Co-efficients. In multiplying together two algebraical expressions where all the powers of x are present, a good deal of labour may be saved by merely writing the co-efficients, and multiplying them together in the ordinary way and inserting the powers of x in the product.

Ex. Multiply $2x^3+x^2-3x+1$ by x^3-2x^2-x+3.

Write only the co-efficients and multiply thus :—

$$\begin{array}{r}
2+1-3+1 \\
1-2-1+3 \\
\hline
2+1-3+1 \\
-4-2+6-2 \\
-2-1+3-1 \\
6+3-9+3 \\
\hline
2-3-7+12+4-10+3
\end{array}$$

The product is $2x^6-3x^5-7x^4+12x^3+4x^2-10x+3$.

A similar mode of operation may be made in Division.

Ex Divide $3x^6-10x^5+4x^4+7x^3-3x^2+3x+2$ by x^2-2x-1.

$$\begin{array}{l}
1-2-1)3-10+4+7-3+3+2(3-4-1+1-2 \\
\qquad\qquad\;\; 3-\;\;6-3 \\
\qquad\qquad\;\; -4+7+7 \\
\qquad\qquad\;\; -4+8+4 \\
\qquad\qquad\qquad -1+3-3 \\
\qquad\qquad\qquad -1+2+1 \\
\qquad\qquad\qquad\qquad 1-4+3 \\
\qquad\qquad\qquad\qquad 1-2-1 \\
\qquad\qquad\qquad\qquad\; -2+4+2 \\
\qquad\qquad\qquad\qquad\; -2+4+2
\end{array}$$

The quotient is $3x^4-4x^3-x^2+x-2$.

Note.—We may use the same method even in cases where all the powers of x are not present provided we insert the missing powers of x with zero co-efficients.

EXERCISE 73.

Multiply by the *Method of Detached Co-efficients* :—

1. $2x^3+3x^2+4x+5$ by x^2-2x+1
2. $6x^3-x+1$ by $6x^3+x-1$
3. px^2+qx+r by qx^2+rx+p.
4. x^3+2x^2+2x+1 by x^2-x+1
5. x^4-2x^2+1 by x^4+2x^2+1
6. $x^3-2x^2y+2xy^2-y^3$ by x^2-xy+y^2.
7. x^3+x^2+x+1 by x^3+x^2+x+1.
8. $x^3+x^2y+xy^2+y^3$ by $x^2+2xy+y^2$.
9. $1+7x+10x^2$ by $1+11x+30x^2$
10. $\frac{1}{8}x^3+\frac{1}{4}x^2+\frac{1}{2}x+1$ by $\frac{1}{4}x^2-\frac{1}{2}x+1$.
11. $3x^4+6x^3-7x^2+6x-4$ by $2x^3-3x^2+5x-3$
12. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)(a^8-a^4+1)$
13. $(x^3+3x^2+3x+1)(x^3-3x^2+3x-1)$.
14. $(a^4+a^3+a^2+a+1)(a^4-a^3+a^2-a+1)$
$\times(a^8-a^6+a^4-a^2+1)$
15. $(x^4-ax^3+bx^2-cx+d)(x^4+ax^3-bx^2+cx-d)$.

Divide by the *Method of Detached Co-efficients*

16. $3x^4-7x^3+13x^2-7x+6$ by x^2-2x+3.
17. $2x^5+7x^4+20x^3+30x^2+34x+35$ by x^2+2x+5
18. $5x^5-18x^3-8x^2+20x-5$ by x^3+2x^2-3
19. $7x^5+19x^4-6x^3$ by $x+3$.
20. $5x^3-4x^2-3x-90$ by $x-3$.
21. $x^5-5x^4+10x^3-10x^2+5x-1$ by x^2-2x+1.
22. $11x^4-2x^3+14x-539$ by x^2-7.
23. $x^9+x^8+x^7+2x^6-x^4-x^3-2x-1$ by x^4+x^3+x+1.
24. $a^{12}+a^{10}+a^8+a^6+a^4+a^2+1$ by $a^6-a^5+a^4-a^3+a^2-a+1$.
25. $a^8+a^6+a^4+a^2+1$ by $a^4-a^3+a^2-a+1$.
26. $a^3-b^3-a^2b+ab^2$ by a^2+b^2. 27. a^4-81 by $a+3$.
28. $a^4-\frac{5}{4}a^3+\frac{11}{8}a^2-\frac{1}{2}a$ by $a^2-\frac{1}{2}a$.

29. $\frac{1}{8}a^3 - \frac{1}{4}a^2b + \frac{1}{6}ab^2 - \frac{1}{27}b^3$ by $\frac{1}{2}a - \frac{1}{3}b$.

30. $10x^{10} + 10x^6 + 10x^3 - 100$ by $x^7 + x^3 - x + 1$.

162. Homogeneity and Symmetry.

Definition: *An expression is said to be homogeneous when the degree of every term in it is the same.*

$Ax + By$ is the form of a *homogeneous* expression of the *first* degree in x and y

$Ax^2 + Bxy + Cy^2$ „ „ „ „ *second* „ „ „

$Ax^3 + Bx^2y + Cxy^2 + Dy^3$ „ „ „ *third* „ „ „

For three quantities x, y, z, the corresponding expressions are :

$Ax + By + Cz$; $Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy$; and

$Ax^3 + By^3 + Cz^3 + mxy^2 + m'x^2y + nyz^2 + n'y^2z + pzx^2$
$+ p'z^2x + qxyz$.

Law of Homogeneity. *The product of two homogeneous expressions, of the m^{th} and n^{th} degrees respectively, is a homogeneous expression of the $(m+n)^{th}$ degree.*

Corollary: The factors of a homogeneous expression are also homogeneous.

Thus $(x+y+z)(x^2+y^2+z^2-xy-yz-zx) = x^3+y^3+z^3-3xyz$.

$(a+b+c)(a+b-c) = a^2+b^2-c^2+2ab$.

$a^2b + b^2a + b^2c + bc^2 + c^2a + ca^2 + 2abc = (a+b)(b+c)(c+a)$.

This law is of great use in testing the accuracy of algebraical work in Multiplication, Division and Factorization.

Definition: *An expression is said to be symmetrical with respect to certain letters when the interchange of any two of these letters throughout the expression would leave the value of the expression unaltered.*

Symmetrical expressions involving two quantities x and y of the 1st, 2nd and 3rd degrees are :—

$Ax + Ay$; $Ax^2 + Bxy + Ay^2$; $Ax^3 + Bxy^2 + Bx^2y + Ay^3$.

The corresponding expressions involving x, y, z are $Ax + Ay + Az$; $Ax^2 + Bxy + Ay^2 + Bxz + Az^2 + Byz$; $Ax^3 + Ay^3 + Az^3 + B(xy^2 + x^2y + yz^2 + y^2z + zx^2 + z^2x) + Cxyz$.

Note.—The terms that have the same co-efficient are those that are derivable from each other by interchanges of the letters.

Rule of Symmetry.—*The algebraical sum, product, or quotient of two symmetrical expressions is a symmetrical expression.*

Obs. The product of two asymmetrical expressions is not necessarily asymmetrical.

Thus $a+b+c$ and $ab+ac+bc$ being both *symmetrical*, their product $a^2b+b^2a+c^2b+bc^2+a^2c+ac^2+3abc$ is *symmetrical.*

Again, x^2yz and xy^2z^2 are both *asymmetrical*; yet their product $x^3y^3z^3$ is *symmetrical.*

Application of the rule of Symmetry: To find the product of $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$, each of which is symmetrical in x, y, z. The product must be symmetrical in x, y, z. It is plain that the term x^3 occurs with the co-efficient unity, hence y^3 and z^3 must occur with the same co-efficient. Again, the term y^2z has the co-efficient 0, hence also, by the principle of symmetry, the five other terms yz^2, z^2x, zx^2, xy^2, x^2y, belonging to the same group must have the co-efficient 0. Lastly, the term $-xyz$ is obtained by taking x from the first factor, hence, it must occur by taking y, and by taking z, *i.e.* the xyz term must have the co-efficient -3. Therefore,

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=x^3+y^3+z^3-3xyz.$$

Alternating Expressions are those that change their sign merely when any two of the letters are interchanged. Thus $(a-b)(b-c)(c-a)$ is an *alternating expression.*

The product or quotient of two alternating expressions involving the same set of letters is a *symmetrical* expression.

The Σ and the π Notation for Symmetrical Expressions.

The symbol $\Sigma\, a$ (read *sigma a*) denotes the *sum* of all the terms of which a is the type; if we are dealing with three letters a, b, c, then $\Sigma\, a=a+b+c$; $\Sigma\, ab=ab+bc+ca$; $\Sigma\, a^2=a^2+b^2+c^2$; $\Sigma\, a^2b=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$; &c.

The symbol $\pi\,(a+b)$ [read *pi* $a+b$] denotes the *product* of all the expressions of which $a+b$ is the type.

If we are dealing with three letters a, b, c, then $\pi(a+b)=(a+b)(b+c)(c+a)$; $\pi\left(\dfrac{a^2}{b+c}\right)=\dfrac{a^2}{b+c}\cdot\dfrac{b^2}{c+a}\cdot\dfrac{c^2}{a+b}$; &c.

163. Method of Indeterminate Co-efficients. In Chapter XIV this method has been explained, and its application illus-

trated by several examples. We shall give some more examples of its application combining the general principles of homogeneity and symmetry.

1. The product of $(x+y)(x+y)$ will be a homogeneous symmetrical expression of the second degree; therefore $(x+y)^2 = px^2+qxy+py^2$. We have to determine p and q. Since the identity holds for *all* values of x and y, it must hold when $x=1$ and $y=0$, therefore $1=p+0+0$; $\therefore p=1$.

We now have $(x+y)^2 = x^2+qxy+y^2$; this must hold when $x=1$ and $y=-1$; therefore $0=1-q+1$, or $q=2$.

Hence $(x+y)^2 = x^2+2xy+y^2$.

2. The product of $(x+y+z)$ $(x^2+y^2+z^2-xy-yz-zx)$ $=A(x^3+y^3+z^3)+B(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+Cxyz$ by the principles of homogeneity and symmetry.

Putting $x=1$, $y=0$, $z=0$, we get $A=1$. Using this value of A, and putting $x=1$, $y=1$, $z=0$, we get $B=0$. Using these values of A and B, and putting $x=1$, $y=1$, $z=1$, we get $C=-3$.

Hence $(x+y+z)$ $(x^2+y^2+z^2-xy-yz-zx)=x^3+y^3+z^3-3xyz$.

3. Resolve into factors $a^3(b-c)+b^3(c-a)+c^3(a-b)$.. (A).

Since (A) vanishes when $a=b$, and also when $b=c$, and also when $c=a$; $\therefore$ it is divisible by $(a-b)(b-c)(c-a)$.

$\therefore$ $(A)=(a-b)(b-c)(c-a)\times B$ where B is a factor of one dimension symmetrical with regard to a, b, c. Therefore (A) $=(a-b)(b-c)(c-a)(a+b+c)p$; where p is a numerical factor to be determined. We have $a^3(b-c)+b^3(c-a)+c^3(a-b)$ $=(a-b)(b-c)(c-a)(a+b+c)p$. Equating the co-efficients of a^3, we have $-(b-c)p=b-c$. $\therefore p=-1$.

$\therefore$ the expression $(A)=-(a-b)(b-c)(c-a)(a+b+c)$.

4. Resolve into factors $(a+b)^5-a^5-b^5$ ....(A).

Since (A) vanishes when $a=0$, and also when $b=0$, and also when $a=-b$; $\therefore$ it is divisible by $ab(a+b)$. Therefore $(A)=ab(a+b)\times B$; where B is a factor of two dimensions symmetrical with regard to a and b. Hence $(A)=ab(a+b)$ $\times(pa^2+qab+pb^2)$ where p and q are numerical quantities to be determined.

We have $(a+b)^5-a^5-b^5=ab(a+b)(pa^2+qab+pb^2)$.

Putting $a=1$ and $b=1$, $32-2=2(2p+q)$, *i.e.*, $2p+q=15$..(1).

Putting $a=2$ and $b=-1$, $1-32+1=-2(5p-2q)$,

i.e., $5p-2q=15$...(2).

Solving these simultaneous equations, we get $p=q=5$.

Hence $(a+b)^5-a^5-b^5=5ab(a+b)(a^2+ab+b^2)$.

EXERCISE 74.

Prove the following identities :—

1. $(a+b+c)^3=\Sigma a^3+3\pi(a+b)$.
2. $\Sigma a\times\Sigma a^2=\Sigma a^3+\Sigma a^2b$.
3. $(b+c-a)(c+a-b)(a+b-c)=\Sigma a^2(b+c)-\Sigma a^3-2abc$.
4. $\Sigma(ca-b^2)(ab-c^2)=(\Sigma bc)(\Sigma bc-\Sigma a)^2$.
5. $4\Sigma(b-c)(b+c-2a)^2=9\Sigma(b-c)(b+c-a)^2$.
6. $\Sigma(b+c-2a)^3=3\pi(b+c-2a)$.
7. $(y+z)^2(z+x)^2(x+y)^2=\Sigma x^4(y+z)^2+2(\Sigma yz)^3-2x^2y^2z^2$.
8. $\Sigma(x+y-z)\{(y-z)^2-(z-x)(x-y)\}=\Sigma x^3-3xyz$.
9. $\Sigma a(\Sigma a^2+\Sigma bc)+\Sigma a\times\Sigma a^2-\Sigma(b+c)^3=3abc$.
10. $\Sigma(a+b)^3-3\pi(a+b)=2\Sigma a^3-6abc$.

Find the factors of :—

11. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.
12. $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$.
13. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
14. $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc$.
15. $(ab+ac+bc)^3-a^3b^3-a^3c^3-b^3c^3$.
16. $(x+y+z)^5-x^5-y^5-z^5$.
17. $a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$.
18. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
19. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$.
20. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.
21. If $5x^2+19x+18=l(x-2)(x-3)+m(x-3)(x-1)$ $+n(x-1)(x-2)$, for *all* values of x, find l, m, n.

22. If $A(x-3)(x-5)+B(x-5)(x-7)+C(x-7)(x-3)=8x-120$, for *all* values of x, find A, B, C.

164. Elimination. Take the two equations $ax+b=0$; and $a_1x+b_1=0$. If these two equations are satisfied by the *same* value of x, then $-\dfrac{b}{a}=-\dfrac{b_1}{a_1}$; *i.e.*, $a_1b=b_1a$. This relation is called the *Eliminant* of the two equations.

Similarly, if we have *three* equations involving *two* unknown quantities and if they are satisfied by the *same* values of the unknown quantities, we can find a relation between the other symbols involved, and the process of finding this relation is called the *Elimination* of the unknown quantities from the given equations.

Ex. 1. Eliminate x from the equations $ax^2+bx+c=0$, $a_1x^2+b_1x+c_1=0$.

If α be the value of x which satisfies both the equations, then we must have—

$$\left.\begin{aligned} a\alpha^2+b\alpha+c&=0\\ a_1\alpha^2+b_1\alpha+c_1&=0\end{aligned}\right\}.$$ Hence, by the method of cross-multiplication, $\dfrac{\alpha^2}{bc_1-b_1c}=\dfrac{\alpha}{a_1c-ac_1}=\dfrac{1}{ab_1-a_1b}$.

$\therefore \dfrac{bc_1-b_1c}{ab_1-a_1b}=\left(\dfrac{a_1c-ac_1}{ab_1-a_1b}\right)^2$. $\therefore (a_1c-ac_1)^2=(ab_1-a_1b)\times(bc_1-b_1c)$, which is the required *Eliminant.*

Ex. 2. Eliminate x and y from $ax+by+c=0$, $a_1x+b_1y+c_1=0$ and $x^2+y^2=1$.

From the first two, by cross-multiplication,

$$\frac{x}{bc_1-b_1c}=\frac{y}{ca_1-c_1a}=\frac{1}{ab_1-a_1b}.\ \therefore x=\frac{bc_1-b_1c}{ab_1-a_1b};\ y=\frac{ca_1-c_1a}{ab_1-a_1b}.$$

Substituting these values in the third, we have—

$$\left(\frac{bc_1-b_1c}{ab_1-a_1b}\right)^2+\left(\frac{ca_1-c_1a}{a_1b-a_1b}\right)^2=1 \text{ or } (bc_1-b_1c)^2+(ca_1-c_1a)^2$$
$=(ab_1-a_1b)^2$, which is the required *Eliminant.*

Ex. 3. Eliminate x, y, z between the equations $x^2+y^2 = cxy$, $y^2+z^2 = ayz$, $z^2+x^2 = bzx$.

We have $\frac{x}{y}+\frac{y}{x}=c$, $\frac{y}{z}+\frac{z}{y}=a$, $\frac{z}{x}+\frac{x}{z}=b$.

By multiplying together these equations, we get—

$$abc=\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+2.$$

$$\therefore\ abc=\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2-4.$$

$\therefore\ abc=c^2+a^2+b^2-4$, which is the required *Eliminant*.

Ex. 4. Eliminate x and y from $x^2-y^2=ax-by$, $4xy = bx+ay$ and $x^2+y^2=1$.

From (1), $x^3-xy^2=ax^2-bxy$
From (2), $4xy^2=bxy+ay^2$

$\therefore\ x^3+3xy^2=a(x^2+y^2)=a.\quad \because\ x^2+y^2=1.$

Similarly $b=y^3+3x^2y$.

$\therefore\ a+b=(x+y)^3$ and $a-b=(x-y)^3$.

$\therefore\ (a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=(x+y)^2+(x-y)^2=2(x^2+y^2).$

$\therefore\ (a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2$, which is the required *Eliminant*.

Ex. 5. Eliminate x, y, z from $\frac{y+z}{y-z}=a$, $\frac{z+x}{z-x}=b$, $\frac{x+y}{x-y}=c$.

From (1), $\frac{y+z+y-z}{y+z-y+z}=\frac{a+1}{a-1}$. $\quad\therefore\ \frac{y}{z}=\frac{a+1}{a-1}$.

Similarly from (2), $\frac{z}{x}=\frac{b+1}{b-1}$ and from (3), $\frac{x}{y}=\frac{c+1}{c-1}$.

$\therefore\ \frac{y}{z}\times\frac{z}{x}\times\frac{x}{y}=\frac{a+1}{a-1}\times\frac{b+1}{b-1}\times\frac{c+1}{c-1}$.

$\therefore\ (a+1)(b+1)(c+1)=(a-1)(b-1)(c-1)$, which is the required *Eliminant*.

Ex. 6. Eliminate x, y, z, from $x^2(y+z)=a^2$,
$y^2(z+x)=b^2$, $z^2(x+y)=c^2$, $xyz=abc$.

Multiplying the first three equations, we have—

$x^2y^2z^2(x+y)(y+z)(z+x)=a^2b^2c^2$; but $x^2y^2z^2=a^2b^2c^2$.

$\therefore\ (x+y)(y+z)(z+x)=1$ ....(A)

Again, adding the first three equations, we have—

$x^2(y+z)+y^2(x+z)+z^2(x+y)=a^2+b^2+c^2$ and $2xyz=2abc$.

$\therefore\ x^2(y+z)+y^2(x+z)+z^2(x+y)+2xyz=a^2+b^2+c^2+2abc$.

$\therefore\ (x+y)(y+z)(z+x)=a^2+b^2+c^2+2abc$ ....(B)

$\therefore$ from (A) and (B), $a^2+b^2+c^2+2abc=1$.

Ex. 7. Eliminate x, y, z from the equations

$$\frac{y}{z}-\frac{z}{y}=a,\ \frac{z}{x}-\frac{x}{z}=b,\ \frac{x}{y}-\frac{y}{x}=c.$$

$$\text{Adding, } a+b+c=\frac{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}{xyz}$$

$$=\frac{(x-y)(y-z)(z-x)}{xyz}.$$

$$\text{Similarly } a-b-c=\frac{x(y^2-z^2)-y(z^2-x^2)-z(x^2-y^2)}{xyz}$$

$$=\frac{(x+y)(z+x)(y-z)}{xyz}.$$

$$b-a-c=\frac{(x+y)(y+z)(z-x)}{xyz}$$

$$c-a-b=\frac{(y+z)(z+x)(x-y)}{xyz}.$$

$$\therefore\ (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

$$=-\frac{(y^2-z^2)^2(z^2-x^2)^2(x^2-y^2)^2}{x^4y^4z^4}$$

$$=-\left(\frac{y}{z}-\frac{z}{y}\right)^2\left(\frac{z}{x}-\frac{x}{z}\right)^2\left(\frac{x}{y}-\frac{y}{x}\right)^2=-a^2b^2c^2.$$

$\therefore\ (a+b+c)(b+c-a)(c+a-b)(a+b-c)+a^2b^2c^2=0$ wh is the required *Eliminant.*

EXERCISE 75.

Eliminate x from the equations :—

1. $\left.\begin{array}{r} ax^2-bx+c=0 \\ bx+a=0 \end{array}\right\}$.   2. $\left.\begin{array}{r} px^2-q=0 \\ rx^3-s=0 \end{array}\right\}$.

3. $\left.\begin{array}{l} ax^2+bx+c=0 \\ px^2+qx+r=0 \end{array}\right\}.$

4. $\left.\begin{array}{l} x^3+\dfrac{3}{x}=4(p^3+q^3) \\ 3x+\dfrac{1}{x^3}=4(p^3-q^3) \end{array}\right\}$

5. $\left.\begin{array}{l} ax^3-b=0 \\ px^4-q=0 \end{array}\right\}.$

6. $\left.\begin{array}{r} lx^2+mx+n=0 \\ x+a=0 \end{array}\right\}.$

7. $\left.\begin{array}{l} x+\dfrac{1}{x}=p \\ x-\dfrac{1}{x}=q \end{array}\right\}.$

8. $\left.\begin{array}{l} px+\dfrac{q}{x}=a \\ px-\dfrac{q}{x}=b \end{array}\right\}.$

9. $\left.\begin{array}{r} ax^3+bx+c=0 \\ a_1x^3+b_1x+c_1=0 \end{array}\right\}.$

10. $\left.\begin{array}{l} px^3+qx^2+r=0 \\ lx^3+mx^2+n=0 \end{array}\right\}.$

11. $ax+A=bx+B=cx+C.$

12. $\dfrac{a}{x^2+a^2}=\dfrac{2b}{x^2+b^2}=\dfrac{4c}{x^2+c^2}$

Eliminate x and y from the equations :—

13. $x+y=a$, $x^3+y^3=b^3$, $x^5+y^5=c^5$.

14. $\dfrac{x^3-a^3}{y^3-b^3}=\dfrac{2a+3b}{3a+2b}$, $x^3-y^3=(a-b)^3$, $x^{\frac{3}{2}}+y^{\frac{3}{2}}=c^{\frac{3}{2}}$.

15. $ax+by=0$, $x+y+xy=0$, $x^2+y^2=1$.

16. $4(x^2+y^2)=ax+by$, $2(x^2-y^2)=ax-by$, $xy=c^2$.

17. $ax^2+b^2y=ay^2+b^2x=c^3$, $x+y=c$.

18. $x-y=a$, $x^2-y^2=b^2$, $x^3-y^3=c^3$.

19. $x+y=a$, $x^2+y^2=b^2$, $x^4+y^4=c^4$.

20. $p(x+y)=q$, $x-y=k(1+xy)$, $xyp=r$.

Eliminate x, y, z from the equations :—

21. $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$, $\dfrac{z}{x+y}=c$.

22. $\dfrac{ax}{by+cz}=\dfrac{by}{cz+ax}=\dfrac{z}{x+y}=\frac{1}{2}$.

23. $x+y+z=p$, $2(xy+yz+zx)=q^2$, $x^3+y^3+z^3=r^3$, $3xyz=s^3$.

24. $x^2(y-z)=a,\ y^2(z-x)=b,\ z^2(x-y)=c,\ xyz=d.$

25. $ay+bx=z,\ az+cx=y,\ bz+cy=x.$

26. $(x+y)^2=4xyc^2,\ (y+z)^2=4yza^2,\ (z+x)^2=4zxb^2.$

27. $(x-y)z^2=c^3,\ (y-z)x^2=a^3,\ (z-x)y^2=b^3,\ (x-y)(y-z)\times(z-x)=3abc.$

28. $\dfrac{x^2(y+z)}{a^3}=\dfrac{y^2(z+x)}{b^3}=\dfrac{z^2(x+y)}{c^3}=\dfrac{xyz}{abc}=1$

29. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{y}{b}+\dfrac{b}{y}=\dfrac{z}{c}+\dfrac{c}{z}$, $xyz=abc$ and $x^2+y^2+z^2+2(ab+ac+bc)=0.$

30. $(x+y-z)(x+z-y)=ayz,\ (x+z-y)(y+z-x)=cxy,\ (y+z-x)(x+y-z)=bxz.$

165. Miscellaneous Examples worked out.

1. If $x(b-c)+y(c-a)+z(a-b)=0$, then will

$$\frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

We have $x(b-c)+y(c-a)+z(a-b)=0$
Identically, $a(b-c)+b(c-a)+c(a-b)=0$ $\Big\}$ $\therefore$ by cross multiplication, $\dfrac{b-c}{cy-bz}=\dfrac{c-a}{az-cx}=\dfrac{a-b}{bx-ay}.$

$$\therefore \frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

2. If $x=cy+bz$, $y=az+cx$ and $z=bx+ay$; shew that $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}.$

From the given equations, we have

$$\left.\begin{aligned} x-cy-bz&=0\ldots\ldots\ldots(1)\\ cx-y+az&=0\ldots\ldots\ldots(2)\\ bx+ay-z&=0\ldots\ldots\ldots(3)\end{aligned}\right\}$$

$\therefore$ by cross-multiplication from (1) and (2),

$$\frac{x}{-ac-b}=\frac{y}{-bc-a}=\frac{z}{-1+c^2};\ \text{or}\ \frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}\ \ldots(A)$$

From (2) and (3), $\dfrac{x}{1-a^2}=\dfrac{y}{ab+c}=\dfrac{z}{ac+b}$ ....(B)

From (1) and (3), $\dfrac{x}{ab+c}=\dfrac{y}{1-b^2}=\dfrac{z}{bc+a}$ ....(C)

From (A) and (B), $\dfrac{x^2}{(1-a^2)(ac+b)}=\dfrac{z^2}{(1-c^2)(ac+b)}$.

$$\therefore \frac{x^2}{1-a^2}=\frac{z^2}{1-c^2}.$$

From (B) and (C), $\dfrac{x^2}{(1-a^2)(ab+c)}=\dfrac{y^2}{(1-b^2)(ab+c)}$.

$\therefore \dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}$. Hence $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

3. If $x+y+z=0, ax+by+cz=0, (b-c)x+(c-a)y+(a-b)z=0$, then shew that $a^2+b^2+c^2-ab-bc-ca=0$.

We have $\left.\begin{array}{r} x+y+z=0 \\ ax+by+cz=0 \end{array}\right\}$ $\therefore$ by cross-multiplication, $\dfrac{x}{c-b}=\dfrac{y}{a-c}=\dfrac{z}{b-a}=k$ (suppose).

Substituting the values of x, y, z in the third equation,

$$-k\{(b-c)^2+(c-a)^2+(a-b)^2\}=0.$$

$\therefore$ $(b-c)^2+(c-a)^2+(a-b)^2=0$. $\therefore$ $a^2+b^2+c^2-ab-ac-bc=0$.

4. If $\dfrac{a-m}{c}+\dfrac{a-n}{d}=\dfrac{1}{b}$, $\dfrac{b-m}{c}+\dfrac{b-n}{d}=\dfrac{1}{a}$, shew that $\dfrac{x-m}{c}+\dfrac{x-n}{d}=\dfrac{x}{ab}$.

Subtracting the second from the first relation, we have $\dfrac{a-b}{c}+\dfrac{a-b}{d}=\dfrac{1}{b}-\dfrac{1}{a}=\dfrac{a-b}{ab}$. $\therefore \dfrac{1}{c}+\dfrac{1}{d}=\dfrac{1}{ab}$ ....(1)

Adding the given relations, $\dfrac{a+b-2m}{c}+\dfrac{a+b-2n}{d}=\dfrac{1}{b}+\dfrac{1}{a}=\dfrac{a+b}{ab}$; but from (1), $\dfrac{a+b}{c}+\dfrac{a+b}{d}=\dfrac{a+b}{ab}$.

$\therefore -\dfrac{2m}{c}-\dfrac{2n}{d}=0$, *i.e.*, $-\dfrac{m}{c}-\dfrac{n}{d}=0$..............................(2)

Again from (1), $\frac{x}{c}+\frac{x}{d}=\frac{x}{ab}$..................................(3)

Adding (2) and (3), $\frac{x-m}{c}+\frac{x-n}{d}=\frac{x}{ab}$.

5. If $x+y=1$, $ax+by=c$, $a^2x+b^2y=c^2$, then $a^nx+b^ny=c^n$.

We have $\left.\begin{aligned} x+y-1&=0 \\ ax+by-c&=0 \end{aligned}\right\}$ ∴ by cross-multiplication, $\frac{x}{b-c}=\frac{y}{c-a}=\frac{1}{b-a}$ .... (1)

Again $\left.\begin{aligned} x+y-1&=0 \\ a^2x+b^2y-c^2&=0 \end{aligned}\right\}$. $\therefore \frac{x}{b^2-c^2}=\frac{y}{c^2-a^2}=\frac{1}{b^2-a^2}$ ......(2)

From (1) and (2), $\frac{b-c}{b-a}=\frac{b^2-c^2}{b^2-a^2}$ and $\frac{c-a}{b-a}=\frac{c^2-a^2}{b^2-a^2}$.

$$\therefore b+a=b+c \text{ and } b+a=c+a. \qquad \therefore a=b=c.$$

$$\therefore a^nx+b^ny=c^nx+c^ny=c^n(x+y)=c^n \quad (\because x+y=1).$$

6. If $x=a(y+z)$, $y=b(z+x)$, $z=c(x+y)$, then

$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ac)}=\frac{z^2}{c(1-ab)}$$

We have $x-ay-az=0$.........(1).

$bx-y+bz=0$.........(2).

$cx+cy-z=0$.........(3).

From (1) and (2), $\frac{x}{-ab-a}=\frac{y}{-ab-b}=\frac{z}{-1+ab}$.

$$\therefore \frac{x}{a(1+b)}=\frac{y}{b(1+a)}=\frac{z}{1-ab} \text{ .... } (A)$$

From (2) and (3), $\frac{x}{1-bc}=\frac{y}{b(1+c)}=\frac{z}{c(1+b)}$ .... (B)

From (1) and (3), $\frac{x}{a(1+c)}=\frac{y}{1-ac}=\frac{z}{c(1+a)}$ ....(C)

From (A) and (B), $\frac{x^2}{a(1+b)(1-bc)}=\frac{z^2}{c(1+b)(1-ab)}$

$$\therefore \frac{x^2}{a(1-bc)}=\frac{z^2}{c(1-ab)}.$$

From (B) and (C), $\dfrac{x^2}{a(1+c)(1-bc)}=\dfrac{y^2}{b(1+c)(1-ac)}$.

$$\therefore \quad \frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ac)}.$$

Hence $\dfrac{x^2}{a(1-bc)}=\dfrac{y^2}{b(1-ac)}=\dfrac{z^2}{c(1-ab)}$.

7. If $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$,

then $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

Identically $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2+(ay-bx)^2$
$+(az-cx)^2+(bz-cy)^2$;

but $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$.

$\therefore$ $(ay-bx)^2+(az-cx)^2+(bz-cy)^2=0$. Here the sum of three positive quantities is zero Therefore each of them must $=0$.

Hence $(ay-bx)^2=0$. $\therefore$ $ay-bx=0$. $\therefore$ $ay=bx$. $\therefore$ $\dfrac{y}{b}=\dfrac{x}{a}$.

$(az-cx)^2=0$. $\therefore$ $az-cx=0$. $\therefore$ $az=cx$. $\therefore$ $\dfrac{z}{c}=\dfrac{x}{a}$.

$(bz-cy)^2=0$. $\therefore$ $bz-cy=0$. $\therefore$ $bz=cy$. $\therefore$ $\dfrac{z}{c}=\dfrac{y}{b}$.

$$\therefore \quad \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

If the sum of the squares of two or more real quantities be zero, then each of the quantities must $=0$.

8. If $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a(x^2-y^2)$; and
$xy-\sqrt{(1-x^2)(1-y^2)}=b(x^2+y^2-1)$; shew that $\dfrac{1}{a^2}+\dfrac{1}{b^2}=1$.

We have $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a(x^2-y^2)$..................(1)

Identically, $x^2(1-y^2)-y^2(1-x^2)=x^2-y^2$..................(2)

Dividing (2) by (1), $x\sqrt{1-y^2}-y\sqrt{1-x^2}=\dfrac{1}{a}$.......(3)

Again, $xy-\sqrt{(1-x^2)(1-y^2)}=b(x^2+y^2-1)$..............(*A*)

Identically, $x^2y^2-(1-x^2)(1-y^2)=x^2+y^2-1$............(*B*)

Dividing (*B*) by (*A*), $xy+\sqrt{(1-x^2)(1-y^2)}=\dfrac{1}{b}$..........(*C*)

Squaring (3) and (*C*) and adding them, we have—

$$x^2(1-y^2)+y^2(1-x^2)-2xy\sqrt{(1-x^2)(1-y^2)}+x^2y^2$$
$$+(1-x^2)(1-y^2)+2xy\sqrt{(1-x^2)(1-y^2)}=\frac{1}{a^2}+\frac{1}{b^2}.$$

Reducing, we get $\dfrac{1}{a^2}+\dfrac{1}{b^2}=1$.

9 If $a+b+c=0$, then $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\times\dfrac{a^2+b^2+c^2}{2}$

Since $a+b+c=0$, $\therefore a=-(b+c)$. $\therefore a^5=-(b+c)^5$

$$\therefore a^5+b^5+c^5=-5b^4c-10b^3c^2-10b^2c^3-5bc^4.$$
$$=-5bc(b^3+2b^2c+2bc^2+c^3)$$
$$=-5bc\{(b^3+c^3)+2bc(b+c)\}$$
$$=-5bc(b+c)(b^2+c^2+bc)$$
$$=5abc\left\{\frac{2b^2+2c^2+2bc}{2}\right\}. \quad [\because b+c=-a].$$
$$\therefore \frac{a^5+b^5+c^5}{5}=abc\left\{\frac{(b+c)^2+b^2+c^2}{2}\right\}$$
$$=\frac{a^3+b^3+c^3}{3}\times\frac{a^2+b^2+c^2}{2}.$$

[$\because a^3+b^3+c^3=3abc$ and $(b+c)^2=a^2$].

10. If $x+y+z=1$, $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$, $ax+by+cz=a+b+c$, find the value of $a^2x+b^2y+c^2z$.

From the first two, $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=(x+y+z)\times\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

$$\therefore x\left(\frac{1}{b}+\frac{1}{c}\right)+y\left(\frac{1}{a}+\frac{1}{c}\right)+z\left(\frac{1}{a}+\frac{1}{b}\right)=0.$$

$\therefore\ a(b+c)x+b(a+c)y+c(a+b)z=0\ldots\ldots\ldots(A)$

From the third, $(a+b+c)(ax+by+cz)=(a+b+c)^2$

$$\therefore\ a^2x+b^2y+c^2z+a(by+cz)+b(ax+cz)+c(ax+by)=(a+b+c)^2.$$

$\therefore\ a^2x+b^2y+c^2z+a(b+c)x+b(a+c)y+c(a+b)z = (a+b+c)^2$, but from (A), $a(b+c)x+b(a+c)y+c(a+b)z=0$.

$\therefore\ a^2x+b^2y+c^2z=(a+b+c)^2.$

1. If $\dfrac{x}{l(mb+nc-la)}=\dfrac{y}{m(nc+la-mb)}=\dfrac{z}{n(la+mb-nc)}$,

that $\dfrac{l}{x(by+cz-ax)}=\dfrac{m}{y(cz+ax-by)}=\dfrac{n}{z(ax+by-cz)}$.

We have $\dfrac{\frac{x}{l}}{mb+nc-la}=\dfrac{\frac{y}{m}}{nc+la-mb}=\dfrac{\frac{z}{n}}{la+mb-nc}$

$$\therefore\ \text{each}=\frac{\frac{x}{l}+\frac{y}{m}}{2nc}=\frac{\frac{y}{m}+\frac{z}{n}}{2la}=\frac{\frac{z}{n}+\frac{x}{l}}{2mb}.$$

$$\therefore\ \frac{mx+ly}{c}=\frac{ny+mz}{a}=\frac{lz+nx}{b}.$$

$$\therefore\ \frac{mxz+lyz}{cz}=\frac{nxy+mxz}{ax}=\frac{lzy+nxy}{by}.$$

$$\therefore\ \text{each}=\frac{mxz+lyz+nxy+mxz-lzy-nxy}{cz+ax-by}\ \text{or}\ \frac{2mxz}{ax+cz-by}.$$

Similarly each $=\dfrac{2nxy}{ax+by-cz}$ and $=\dfrac{2lyz}{by+cz-ax}$.

$\therefore\ \dfrac{lyz}{by+cz-ax}=\dfrac{mxz}{ax+cz-by}=\dfrac{nxy}{ax+by-cz}$. Divide each by xyz.

$$\therefore\ \frac{l}{x(by+cz-ax)}=\frac{m}{y(ax+cz-by)}=\frac{n}{z(ax+by-cz)}$$

2. Solve $4^{1+x}+4^{1-x}=10$. We have $4\,.\,4^x+4\,.\,4^{-x}=10$.

$\therefore\ 2.4^x+\dfrac{2}{4^x}=5 \qquad \therefore\ 2.\,4^{2x}-5\,.\,4^x+2=0.$

$\therefore\ 4^x=\frac{1}{4}(5\pm 3)=2$ or $\frac{1}{2}$. $\qquad \therefore\ 2^{2x}=2^1$ or 2^{-1}.

$\therefore\ 2x=1$ or -1. $\qquad \therefore\ x=\frac{1}{2}$ or $-\frac{1}{2}$.

13. Solve $x^{x+y}=y^4$ and $y^{x+y}=x^a$.

From the first, $x^{\frac{x+y}{4a}}=y$ and from the 2nd, $x^{\frac{a}{x+y}}=y$.

$\therefore\ x^{\frac{x+y}{4a}}=x^{\frac{a}{x+y}}$. $\therefore \frac{x+y}{4a}=\frac{a}{x+y}$. $\therefore\ (x+y)^2=4a^2$.

$\therefore\ x+y=2a$. Substituting this value in the first, $x^2=y^4$

$\therefore\ x=y^2$. $\therefore\ x+y=y^2+y=2a$.

$\therefore\ y^2+y-2a=0$. $\therefore\ y=\frac{1}{2}(-1\pm\sqrt{8a+1})$.

$\therefore\ x=2a-\frac{1}{2}(-1\pm\sqrt{8a+1})=\frac{1}{2}(4a+1\mp\sqrt{8a+1})$.

14. If x be a *proper* fraction, shew that—

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots\ldots}=(1+x)(1+x^2)(1+x^3)\ldots\ldots\ldots\ldots$$

Let $(1-x)(1-x^3)(1-x^5)\ldots\ldots\ldots=a$; $(1+x)(1+x^3)\times(1+x^5)\ldots\ldots=b$,

$(1+x^2)(1+x^4)(1+x^6)\ldots\ldots=c$ and $(1-x^2)(1-x^4)\times(1-x^6)\ldots\ldots=d$.

Then $(1-x^2)(1-x^6)(1-x^{10})\ldots\ldots=ab$ and $(1-x^4)\times(1-x^8)(1-x^{12})=cd$

$\therefore\ (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})\ldots\ldots\ldots=abcd$,
but $(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})\ldots\ldots\ldots\ldots=d$.

$\therefore\ abcd=d$. $\therefore \frac{1}{a}=bc$.

$$\therefore\ \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}=(1+x)(1+x^2)(1+x^3)\times(1+x^4)\ldots\ldots\ldots$$

15. (*a*) The product of the m^{th} and n^{th} terms in the quotient of 1 divided by $1-x$ is $\frac{1}{x^2}$; shew that $m^3+n^3=0$.

We have by division $\frac{1}{1-x}=1+x+x^2+x^3+x^4+$ &c.

It is plain that the m^{th} term $=x^{m-1}$ and the n^{th} term $=x^{n-1}$.

$\therefore$ the product of the m^{th} and n^{th} terms $=x^{m-1}\times x^{n-1}$
$=x^{m+n-2}$

By the question, $x^{m+n-2} = \frac{1}{x^2} = x^{-2}$. $\therefore m+n-2=-2$

$\therefore m+n=0$. $\therefore (m+n)(m^2-mn+n^2)=0$. $\therefore m^3+n^3=0$

(*b*) The quotient obtained by dividing the r^{th} term by the $(2r)^{th}$ term in the quotient of $\frac{1}{1-x}$ is $\frac{1}{x^9}$, find r

$$\frac{1}{1-x} = 1+x+(x)^2+(x)^3+ \ \&c$$

The r^{th} term $=(x)^{r-1}$ and $(2r)^{th}$ term $=(x)^{2r-1}$.

$$\frac{\text{The } r^{th} \text{ term}}{\text{The } (2r)^{th} \text{ term}} = \frac{x^{(r-1)}}{x^{(r^2-1)}} = x^{-r^2} \quad \text{By the question,}$$

$$x^{-r^2} = \frac{1}{x^9} = x^{-9} \quad \therefore -r^2=-9. \quad \therefore r^2=9 \quad \therefore r=3$$

16 If the H.C.F. of $ax^3+bx^2+(a+b)x+1$ and $ax^3+(b+1)x^2+(a+b-1)x+2$ be a *binomial*, shew that $(b+1)\times(a+b-1)=(a+2b)^2$

$$
\begin{array}{l}
ax^3+bx^2+(a+b)x+1 \,\big)\, ax^3+(b+1)x^2+(a+b-1)x+2 \,\big(\, 1 \\
\qquad\qquad\qquad\qquad\quad \underline{ax^3+bx^2+(a+b)x+1} \\
\qquad\qquad\qquad\quad x^2-x+1)\,ax^3+bx^2+(a+b)x+1\,(ax+a+b \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{ax^3-ax^2+ax} \\
\qquad\qquad\qquad\qquad\qquad\quad x^2(a+b)+bx+1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2(a+b)-x(a+b)+a+b} \\
\qquad\qquad\qquad\qquad\qquad\qquad x(a+2b)-(a+b-1)
\end{array}
$$

$$
x(a+2b)-(a+b-1)\,\big)\, x^2-x+1 \,\Big[\, \frac{x}{a+2b} - \frac{1+b}{(a+2b)^2}
$$
$$
\underline{x^2 - \frac{x(a+b-1)}{a+2b}}
$$

Remainder is $x\left\{\frac{a+b-1}{a+2b}-1\right\}+1 = \frac{-x(1+b)}{a+2b}+1$

$$\underline{\frac{-x(1+b)}{a+2b}+\frac{(a+b-1)(1+b)}{(a+2b)^2}}$$

If $x(a+2b)-(a+b-1)$ be the H.C.F., the remainder must vanish. Therefore $\frac{(a+b-1)(1+b)}{(a+2b)^2}=1$

Hence $(a+b-1)(b+1)=(a+2b)^2$

13. Find the *difference* between—
$\{1-(n+1)(n+2)x^n+2n(n+2)x^{n+1}-n(n+1)x^{n+2}\} \div (1-x)^3$
and the same expression when n is *increased* by 1.

14. The value of the expression a^3-3a^2+2 remains unaltered when a is increased by $2\sqrt{3}$. Determine that value.

15. If $\alpha+\beta+\gamma=-\frac{b}{a}$; $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$; $\alpha\beta\gamma=-\frac{d}{a}$ and α, β, γ be in *continued* proportion; shew that $a : d=b^3 : c^3$.

If $\frac{mn}{l}(a-k)=\frac{nl}{m}(b-k)=\frac{lm}{n}(c-k)=la'=mb'=nc'$,

than $\frac{a'}{aa'-b'c'}=\frac{b'}{bb'-c'a'}=\frac{c'}{cc'-a'b'}$.

17. Extract the square root of $(2a^x+a^{-y})(2a^x+a^{-x})$
$+2(2a^{\frac{x+y}{2}}+\frac{1}{2}+a^{\frac{-x+y}{2}})$.

18. If two algebraical expressions and their H.C.F. be of the m^{th}, n^{th} and r^{th} degree in x respectively, no common multiple of A and B can be of less than the $(m+n-r)^{th}$ degree.

19. If $f(x)+f^1(x)=a^x=\frac{1}{f(x)-f^1(x)}$, shew that—
$f(x)f^1(x)+f(y)f^1(y)=f(x-y)f^1(x+y)$.

20. Shew that the equation $(b-x)^2-4(a-x)(c-x)=0$ has *real* roots whatever be the values of a, b, c.

III.

21. If M and N be any two odd numbers ($M>N$), shew that M^2-N^2 is divisible by 8.

22. Substitute $\frac{1}{2}(b+c)$ for x in the expression—
$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-a)(x-c)}{(b-a)(b-c)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}$ and simplify the result.

23. Solve (i) $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12$, $\frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8$, $\frac{x}{2}+\frac{z}{3}=10$.

(ii) $x+a\sqrt{y}=y+a\sqrt{x}=b^2$.

24. A boat which can row 6 miles per hour in still water, takes $3\frac{3}{4}$ hours to go to a place 10 miles down the river and come back. At what rate does the current flow?

25. If $ax+by+cz=a^2x+b^2y+c^2z=\dfrac{x}{b+c-a}+\dfrac{y}{c+a-b}$ $+\dfrac{z}{a+b-c}=0$, prove that $bc(b-c)^3+ca(c-a)^3+ab(a-b)^3=0$.

26. Write down $x^4+\dfrac{97}{6}x^2+1-\dfrac{91}{12}x(x^2+1)$ as the product of *four* factors.

27. Solve (i) $z+y+m(x-y)=a$; $z-x-m(z+x)=b$; and $(1-m)^2yz=ab$. (ii) $\dfrac{x+a}{x}\cdot\dfrac{x+b}{x}\cdot\dfrac{x+c}{x}=1$.

28. If $x+yz=\{(1-a^2)(1-y^2)(1-z^2)\}^{\frac{1}{2}}$, $y+xz$

$=\{(1-b^2)(1-x^2)(1-z^2)\}^{\frac{1}{2}}$, and $z+xy$

$=\{(1-c^2)(1-x^2)(1-y^2)\}^{\frac{1}{2}}$,

shew that $\dfrac{x^2-y^2}{a^2-b^2}=\dfrac{y^2-z^2}{b^2-c^2}=\dfrac{z^2-x^2}{c^2-a^2}$.

29. Simplify $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-c)(b-a)(x-b)}$ $+\dfrac{c}{(c-a)(c-b)(x-c)}$.

30. Find two numbers, of which the product shall be 6 and the sum of their cubes 35.

IV.

31. Divide $1+2x$ by $1-3x$ to 5 terms in the quotient.

32. Solve (i) $\dfrac{x^2+2x-3}{x^2+5x+6}-\dfrac{x}{x+2}=\dfrac{1}{2}$. (ii) $\dfrac{x+1}{x+2}=x^3$.

33. If $\dfrac{1}{a}+\dfrac{1}{c}=\dfrac{2}{b}$, then $\left(a-\dfrac{b}{2}\right)\left(c-\dfrac{b}{2}\right)=\dfrac{b^2}{4}$.

34. If $a+c=2b$, then $\frac{2}{9}(a+b+c)^3=a^2(b+c)+b^2(a+c)$ $+c^2(a+b)$.

35. Find the H.C.F. of $2x^5-8x^4+12x^3-8x^2+2x$ and $3x^5-6x^3+3x$ and the L.C.M. of x^4+x^2-2 and x^3-3x^2-x+3.

36. Solve (i) $\dfrac{1}{a^2-x^2}-a=\dfrac{ax}{a-x}+\dfrac{a}{a+x}$.

(ii) $x^2+x^{-2}=(x-x^{-1})+2$.

37. Find the square root of $7-\sqrt{10}+\sqrt{14}-\sqrt{35}$; and $4+\sqrt{15}$.

38. Resolve (i) $3x^2-14xy+5x-5y^2+7y-2$.

(ii) $6x^2y^2-7xy^3-3y^4$.

39. Eliminate x, y, z from $x+y+z=a$; $x^2+y^2+z^2=b^2$; $x^3+y^3+z^3-3xyz=c^3$.

40. A sets off from Madras to Bangalore and B at the same time from Bangalore to Madras and they travel uniformly. A reaches Bangalore 16 hours and B reaches Madras 36 hours after they have met on the road. Find in what time each has performed the journey.

V.

41. If $\dfrac{x}{2a-b+2c}=\dfrac{y}{2b-c+2a}=\dfrac{z}{2c-a+2b}$,

prove that $\dfrac{2x+2y-z}{a}=\dfrac{2y+2z-x}{b}=\dfrac{2z+2x-y}{c}$.

42. Solve—

(i) $3(x^2+y^2)-2xy=27$; $2(x^2+y^2)-3xy=8$.

(ii) $\dfrac{x+1}{y-1}-\dfrac{x-1}{y}=\dfrac{6}{y}$; $x-y=1$.

(iii) $x+y+z=39$; $x^2+y^2+z^2=741$; $xy=z^2$.

(iv) $\dfrac{y^2z}{2\sqrt{3}}=\dfrac{z^2x}{3}=\dfrac{x^2y}{\sqrt{2}}=1$.

(v) $x+y+z=\dfrac{a^2}{x}=\dfrac{b^2}{y}=\dfrac{c^2}{z}$.

43. If the product of the p^{th} and q^{th} terms in the quotient of $\dfrac{1}{1-x}$ be x^{pq-2}, then $\dfrac{1}{p}+\dfrac{1}{q}=1$.

44. Find $\dfrac{a^2+b^2+2ab-4}{a+b-2}$ and $\dfrac{a^3+b^3+6ab-8}{a+b-2}$ when $a=b=1$.

45. If $(a^2+bc)^2(b^2+ac)^2(c^2+ab)^2=(a^2-bc)^2(b^2-ac)^2$ $\times(c^2-ab)^2$, then $a^3+b^3+c^3+abc=0$.

46. If $x=a(y+z)$, $y=b(x+z)$, $z=c(x+y)$, then $ab+bc+ca+2abc=1$.

47. If $\left(\dfrac{x-a}{a}\right)^2+\left(\dfrac{y-b}{b}\right)^2+\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{1}{2}$

48. Solve (i) $\sqrt{3x+4}+\sqrt{3x-5}=9$.

(ii) $\sqrt{9x}+\sqrt{x}=\sqrt{15x+4}$.

49. Solve (i) $\sqrt{\left(x-\dfrac{1}{x}\right)}+\sqrt{\left(1-\dfrac{1}{x}\right)}=x$.

(ii) $xy=x+y$; $xz=2(x+z)$; $yz=3(y+z)$.

(iii) $\dfrac{2x^2}{\sqrt{x}(15+x)-8x}+\dfrac{123+41\sqrt{x}}{\sqrt{x}(5-\sqrt{x})}=\dfrac{20\sqrt{x}+4x}{3-\sqrt{x}}$.

50. A cask, which held 270 gallons, was filled with a mixture of brandy, wine and water; there were 30 gallons of wine in it more than of brandy and 30 of water more than there were of wine and brandy together. How many were there of each?

VI.

51. If $\dfrac{a+b+c}{b+c+d}=\dfrac{c+d+a}{d+a+b}$, then shew that—

$$(a+b+c)(a+b+d)=\frac{1}{(a+b-c-d)^2}(a-c)(a-d)(b-c)(b-d).$$

52. Find the value of $\dfrac{(a+b)^m-a^m}{b}$, when $b=0$.

53. If $a^2b^2+b^2c^2+c^2a^2=2abc(a+b-c)$, shew that—

(i) $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}$. (ii) $(a+b-c)^2=a^2+b^2+c^2$.

54. If $\dfrac{b^2-c^2}{x-y}=\dfrac{bc}{a}$ and $\dfrac{c^2-a^2}{y-z}=\dfrac{ca}{b}$, then $\dfrac{a^2-b^2}{z-x}=\dfrac{ab}{c}$.

55. If $a\left(\frac{1}{y}+\frac{1}{z}-\frac{1}{x}\right)=b\left(\frac{1}{z}+\frac{1}{x}-\frac{1}{y}\right)=c\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right)=1$, then $a(b+c)x=b(a+c)y=c(a+b)z$.

56. Solve (i) $x^2+y^2=7+xy$; $x^3+y^3=6xy-1$.

(ii) $x^2+\frac{10}{3}x=19$. (iii) $3x^2-20x=5$.

(iv) $x^4-x^2+y^4-y^2=84$; $x^2+x^2y^2+y^2=49$.

57. If x^3+ax^2+bx+c and x^3+dx^2+ex+f have a common quadratic factor, then shew that $\frac{c-f}{a-d}=\frac{dc-af}{b-e}=\frac{ec-bf}{c-f}$.

58. If $c=a\sqrt{1-b^2}+b\sqrt{1-a^2}$, then $a\sqrt{1-a^2}+b\sqrt{1-b^2}+c\sqrt{1-c^2}=2abc$.

59. If $m\sqrt{1-x^2}-nx=\sqrt{m^2+n^2}$ and

$$\frac{x^2}{a^2}+\frac{1-x^2}{b^2}=\frac{1}{m^2+x^2}, \text{ shew that } \frac{m^2}{b^2}+\frac{n^2}{a^2}=1.$$

60. A person has £1,800 which he divides into two portions and lends at different rates of interest so that the two portions produce equal returns. If the first portion had been lent at the second rate of interest, it would have produced £50; and if the second portion had been lent at the first rate of interest, it would have produced £32. Find the rates of interest.

VII.

61. If $qx+py=r$; $rx+pz=q$; $ry+qz=p$; then—
$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2\sqrt{(1-x^2)(1-y^2)(1-z^2)}$

62. Eliminate x and y from $x(a+x)=y(b+y)=x^2+y^2=1$.

63. Eliminate a, b and c from $(a+b)^2=4abx^2$; $(b+c)^2=4bcz^2$ and $(a+c)^2=4acy^2$.

64. Solve (i) $8^{x+2}+2^{3x+2}+2^{3x+1}=112$.

(ii) $\sqrt[m]{b+x}=\sqrt[2m]{x^2+3bx-b^2}$.

(iii) $4(x-1)(x-2)(2x-1)(2x-3)+1=0$.

(iv) $(x+4)(x-5)+\sqrt{x^2-x+1}=0$.

65. If $ax+by+cz=a$, $bx+cy+az=b$ and $cx+ay+bz=c$, shew that $a^3+b^3+c^3=3abc$, if $x+y+z-1$ is not $=0$.

66. When will the hands of a clock be *at right angles* between 1 and 2 o'clock?

67. The G.C.M. of two numbers is 8 and their L.C.M. is 240. How many pairs of numbers are there?

68. If $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x^2}{a}+\frac{y^2}{b}=\frac{ab}{a+b}$, then $\frac{x^3}{a}+\frac{y^3}{b}=\left(\frac{ab}{a+b}\right)^2$ and $\frac{x^{n+1}}{a}+\frac{y^{n+1}}{b}=\left(\frac{ab}{a+b}\right)^n$

69 Sum up $\frac{1}{1+a}+\frac{2a}{1+a^2}+\frac{2^2a^3}{1+a^4}+\dots\dots+\frac{2^{n-1}a^{2^{n-1}-1}}{1+a^{2^{n-1}}}$.

70. The sum of the squares of two numbers is 1450, their G.C.M is 5 and their L.C.M. is 105. Find the numbers.

VIII.

71. Simplify $\frac{a-x}{(a-1)(x-1)}+\frac{b-y}{(b-1)(y-1)}+\frac{c-z}{(c-1)(z-1)}$ $+\frac{x-b}{(x-1)(b-1)}+\frac{y-c}{(y-1)(c-1)}+\frac{z-a}{(z-1)(a-1)}$.

72. Shew that $x\frac{by-cz}{b-c}=y\frac{cz-ax}{c-a}=z\frac{ax-by}{a-b}$ if $\frac{a-b}{z}+\frac{b-c}{x}+\frac{c-a}{y}=0$.

73. If $x+y+z=(a+b)x+(b+c)y+(c+a)z=(b+c)x+(a+c)y+(a+b)z=0$, then $a^2+b^2+c^2=ab+bc+ca$.

74. Find the H.C.F. of x^5+5x^3+6 and $3x^4+120x+117$ and the L.C.M. of x^3+4x^2-5 and $2x^3+x^2-8x+5$.

75. Find how many gallons of water must be mixed with 80 gallons of spirit which cost 15*s.* a gallon, so that by selling the mixture at 12*s.* a gallon there may be a gain of 10 per cent. on the outlay

76. Extract the square root of $28-6\sqrt{3}$ and simplify
$$\frac{a^{\frac{3n}{2}}-a^{-\frac{3n}{2}}}{a^{\frac{n}{2}}-a^{-\frac{n}{2}}}.$$

77. Solve (i) $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{3})(x-\frac{1}{4})=(x-\frac{1}{4})(x-\frac{1}{5})$.

(ii) $(x-y)(x^2+4y^2)=x^3+y^3$; $xy=c^2$.

78. Reduce $\dfrac{2x^3+ax^2+4a^2x-7a^3}{x^3-7ax^2+8a^2x-2a^3}$ to its lowest terms.

79. Eliminate x from $Ax^2+Bx+C=0$ and $A'x^2+B'x+C'=0$.

If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, then $\dfrac{x^2+y^2}{a^2+b^2}=\dfrac{y^2+z^2}{b^2+c^2}=\dfrac{z^2+x^2}{c^2+a^2}$
$$=\left(\frac{lx+my+nz}{la+mb+nc}\right)^2.$$

80. The fore-wheel of a carriage makes 6 more revolutions than the hind wheel in 120 yards; but the former would only make 4 more revolutions than the latter in the same distance if the spokes of each be lengthened by 5 inches; what is the circumference of each wheel?

IX.

81. If $(x^2-y^2)z=(y^2-z^2)x$, then $\dfrac{x}{z}=\dfrac{x^2}{y^2}$.

Eliminate x from $x^3+\dfrac{1}{x^3}+3\left(x+\dfrac{1}{x}\right)=m$ and $x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)=n$.

82. Solve (i) $\dfrac{x-\sqrt{2x+1}}{x+\sqrt{2x+1}}=\dfrac{a}{b}$. (ii) $\dfrac{x+2a}{x-2a}=\left(\dfrac{x+a}{x-a}\right)^3$.

(iii) $x^2+xy=a^2$; $x^2-xy=b^2$.

(iv) $(1+x)(1+y)=10$; $x^2y+xy^2=18$.

83. Shew that if $l_1x+m_1y+n_1z=0$, $l_2x+m_2y+n_2z=0$ and $ax+by+cz=d$, then $\dfrac{x}{m_1n_2-m_2n_1}=\dfrac{y}{n_1l_2-n_2l_1}=\dfrac{z}{l_1m_2-l_2m_1}$
$$=\frac{d}{a(m_1n_2-m_2n_1)+b(n_1l_2-n_2l_1)+c(l_1m_2-l_2m_1)}.$$

84. If $2s = a + b + c$, then $abc - a(s-b)(s-c) - b(s-a)(s-c) - c(s-a)(s-b) = 2(s-a)(s-b)(s-c)$.

85. Find the H.C.F. of $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$, and the L.C.M. of $6x^4 - 7x^2 + 2$ and $2x^3 + 6x^2 - x - 3$.

86. If $2s = a + b + c$, then $(s-a)^3 + (s-b)^3 + (s-c)^3 - 3(s-a) \times (s-b)(s-c) = \frac{1}{2}(a^3 + b^2 + c^3 - 3abc)$.

87. Is $\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{b^2(x-a)(x-c)}{(b-a)(b-c)} + \dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2$ an *equation* or an *identity*?

88. Find the value of $x^4 + 4x^3 + 6x^2 + 4x + 9$, when $x = \sqrt{-2} - 1$. If p be the difference between any fraction and unity, and q the difference between its reciprocal and unity, then $pq = p - q$.

89. Solve (i) $3x - \sqrt{2x^2 + 6x + 1} = 1 - x^2$.

(ii) $x^2 + 2x = 12 - 4\sqrt{x^2 + 2x}$.

(iii) $3x + 2\sqrt{x^2 - 3x + 9} = x^2 + 6$.

(iv) $(4x + 2)^4 + (x + 3)^4 = (2x + 4)^4 + (3x + 1)^4$

(v) $x^2(y + z) = a^3$, $y^2(z + x) = b^3$; $z^2(x + y) = c^3$.

90. Two trains moving in opposite directions pass each other somewhere between two stations A and B. One train leaves B at 10 minutes past 4 and arrives at A 20 minutes to 5. The other leaves A at 10 minutes past 4 and arrives at B at 20 minutes to 5. Find the time at which they meet.

X.

91. Prove that $(x + y)(y + z)(z + x) - x^3 - y^3 - z^3 = 4xyz + (y + z - x)(z + x - y)(x + y - z)$ and

$$\frac{x(1-y^2)(1-z^2) + y(1-x^2)(1-z^2) + z(1-x^2)(1-y^2)}{x + y + z - xyz} = 1 - xy - yz - zx$$

92. Simplify (i) $\dfrac{a^{\frac{1}{3}}+3b^{\frac{1}{3}}}{a^{\frac{1}{3}}-3b^{\frac{1}{3}}}+\dfrac{a^{\frac{2}{3}}-3a^{\frac{1}{3}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}}{a^{\frac{2}{3}}+3a^{\frac{1}{3}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}}$.

(ii) $\left(1-\dfrac{b^2}{a^2}\right)\left(1-\dfrac{ab-b^2}{a^2}\right)\dfrac{a^4}{a^3+b^3}\times\dfrac{a-b}{a^2+b^2}$.

93. Find the H.C.F. and L.C.M. of—

x^3-x^2-4x+4 and $x^3-3x^2-4x+12$.

94. Solve (i) $\dfrac{x-4a}{x-3a}+\dfrac{x-5a}{x-4a}=\dfrac{x+6a}{x-4a}+\dfrac{x+5a}{x-3a}$.

(ii) $\sqrt{\{x+\sqrt{(x^2-4)}\}}=\sqrt{\left(\dfrac{x+2}{2}\right)}+\sqrt{(2x-4)}$

(iii) $x+y+z=18$; $x^2+y^2+z^2=110$; $x(y+z)=65$

(iv) $x\sqrt{y}+y\sqrt{x}=-6$; $x^{\frac{3}{2}}+y^{\frac{3}{2}}=26$.

(v) $x+y+z=xyz=6$; $xy+yz+zx=11$.

95. If $ax=\dfrac{2pq}{1+q^2}$, find $\dfrac{\sqrt{\left(\dfrac{p}{a}+x\right)}+\sqrt{\left(\dfrac{p}{a}-x\right)}}{\sqrt{\left(\dfrac{p}{a}+x\right)}-\sqrt{\left(\dfrac{p}{a}-x\right)}}$.

96. If $y=x-\dfrac{1}{x}$, express $x^5-\dfrac{1}{x^5}$, in terms of y.

97. If $ax+by=a+b$ and $a^2x^2+b^2y^2=a^2+b^2$, then

$a^nx^n+b^ny^n=a^n+b^n$.

98. Solve (i) $(x+4)(x-3)+\sqrt{\{(x+3)(x-2)\}}=36$.

(ii) $\dfrac{mx}{mx+a}+\dfrac{nx}{nx+a}=2\dfrac{px}{px+a}$.

(iii) $x^2+y^2+z^2=2a^2$; $x+y+z=2b$; $yz=c^2$.

(iv) $ax+cy+bz=cx+by+az=bx+ay+cz$

$=a^3+b^3+c^3-3abc$

99. Resolve into the product of two *linear* factors the quadratic exqressions $7x^2-3x-160$ and $mnx^2+(n^2-m^2)x-mn$.

100. A man starts from the foot of a mountain to walk to its summit ; his rate of walking during the second half o

the distance is half a mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours by walking at a uniform rate which is one mile per hour more than his rate during the first half of the ascent. Find the distance of the summit and his rate of walking.

XI.

101. If $P = a^2(cx+d)^2\{3c(ax+b) - a(cx+d)\}$ and $Q = c^2(ax+b)^2\{3a(cx+d) - c(ax+b)\}$, find the cube root of $Q-P$.

102. If $A=(b-c)(a-d)$, $B=(c-a)(b-d)$, $C=(a-b)(c-d)$, find in the simplest form $A^3+B^3+C^3-3ABC$.

103. Shew that $a^4+b^4+c^4-(a+b+c)(a^3+b^3+c^3)$
$+(ab+ac+bc)(a^2+b^2+c^2)=abc(a+b+c)$.

104. If $\dfrac{x}{a^2+x^2}=\dfrac{2y}{a^2+y^2}=\dfrac{4z}{a^2+z^2}$, shew that $x(y^2-z^2)$ $+2y(z^2-x^2)+4z(x^2-y^2)=0$.

105. Eliminate l, m and n from $a^2l^3+b^2m^3+c^2n^3=a^2l+b^2m$ $+c^2n$, $al=bm=cn$, $l^2+m^2+n^2=b$.

106. For what value of x will the expression $x^4+8x^3+22x^2$ $+23x+19$ be a *perfect* square? If the two quadratic equations $ax^2+2bx+c=0$ and $a_1x^2+2b_1x+c_1=0$, have a *common* root, prove that $(ac_1+a_1c-2bb_1)^2=4(ac-b^2)(a_1c_1-b_1^2)$.

107. If $P=(a-b)^2+(b-c)^2+(c-a)^2$, express $(a-b)^2$ $\times(a+b-2c)^2+(b-c)^2(b+c-2a)^2+(c-a)^2(c+a-2b)^2$ in terms of P.

108. Prove that $(x+y+z)\{x^2+y^2+z^2-(y-z)^2-(z-x)^2$ $-(x-y)^2\}=8xyz+(y+z-x)(x+z-y)(x+y-z)$.

109. Solve (i) $(a+x)^{\frac{1}{4}}+(a-x)^{\frac{1}{4}}=h$.

(ii) $x^{\frac{1}{4}}+y^{\frac{1}{4}}=6$; $x^{\frac{5}{4}}+y^{\frac{5}{4}}=126$.

(iii) $a(x^2+1)=x(a^2+1)$.

(iv) $\sqrt{(x+y)}=\sqrt{x}+1$; $x-y=7$.

(v) $y-z=2$; $y(x+z)=65$; $z(x+y)=45$.

(vi) $x^2+xy+y^2=28$; $x-y-z=0$; $\dfrac{x}{z}+\dfrac{z}{y}=\frac{3}{8}xz$.

110. A rides at the rate of 8 miles an hour; B walks at the rate of $3\frac{1}{2}$; B starts first, and after a certain interval A sets off to overtake him. When he had ridden 14 miles, his horse broke down, and he had to walk on, which he did at the rate of 4 miles an hour, overtaking B in 14 miles more. What start had B?

XII.

111. If $x+y+z=0$, then $z\dfrac{x^3-y^3}{x-y}+x\dfrac{y^3-z^3}{y-z}+y\dfrac{z^3-x^3}{z-x}=0$ and x^2-4yz is a perfect square.

112. Shew that $(x+y+z)^3-(x+y)^3-(y+z)^3-(z+x)^3+x^3+y^3+z^3=6xyz$.

113. If $2s=a+b+c$ and $2S^2=a^2+b^2+c^2$, shew that $(S^2-a^2)\times(S^2-b^2)+(S^2-b^2)(S^2-c^2)+(S^2-a^2)(S^2-c^2)=4s(s-a)\times(s-b)(s-c)$.

114. Shew that $\dfrac{d^m(a-b)(b-c)+b^m(a-d)(c-d)}{c^m(a-b)(a-d)+a^m(b-c)(c-d)}=\dfrac{b-d}{a-c}$ when $m=1$ or 2.

115. If $ab+bc+ca=1$, shew that $\dfrac{a}{1-a^2}+\dfrac{b}{1-b^2}+\dfrac{c}{1-c^2}$

$$=\frac{4abc}{(1-a^2)(1-b^2)(1-c^2)}.$$

116. Solve (i) $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

(ii) $(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)\times(x-6)(x-10)$.

(iii) $\dfrac{1}{x+12b}+\dfrac{2}{x-6b}+\dfrac{3}{x+4b}=\dfrac{6}{x+2b}$.

117. If $t=\dfrac{2}{2-w}$, $w=\dfrac{2}{2-z}$, $z=\dfrac{2}{2-y}$, $y=\dfrac{2}{2-x}$, find the relation between t and x.

118. Simplify :—

$$\frac{(1-10x^2+5x^4)(5-30x^2+5x^4)+(5x-10x^3+x^5)(20x-20x^3)}{(5x-10x^3+x^5)^2+(1-10x^2+5x^4)^2}.$$

119. Shew that $(a^2+b^2+c^2+d^2)\ (x^2+y^2+z^2+u^2)=(ax-by+cz-du)^2+(ay+bx-cu-dz)^2+(az-bu-cx+dy)^2+(au+bz+cy+dx)^2$.

120. A person leaves £12,670 to be divided among his five children and three brothers, so that after the legacy duty has been paid, each child's share shall be twice as great as each brother's. The legacy duty on a child's share being one per cent., and on a brother's share three per cent.; find what amounts they respectively receive.

XIII.

121. Simplify: $(a+b+c)(x+y+z)+(b+c-a)(z+y-x)+(a+b-c)(x+y-z)+(c+a-b)(z+x-y)$.

122. Find the square root of $x-z+2\sqrt{(xy+yz-zx-y^2)}$.

123. Solve (i) $(3x-1)^2+(4x-2)^2=(5x-3)^2$.

(ii) $\dfrac{x-2a}{x-3a}+\dfrac{y-4b}{y-3b}=2$; $\dfrac{x+2a}{x+a}=\dfrac{y+5b}{y+3b}$.

(iii) $x^3+y^3+z^3=3xyz$; $x-a=y-b=z-c$.

124. Resolve $2x^2-21xy-11y^2-x+34y-3$ into *rational* factors of the *first* degree.

125. If $a+b+c=0$, shew that the three following expressions have a common binomial factor and find it.

$$ax^2+bx+c,\ cx^2+ax+b \text{ and } bx^2+cx+a.$$

126. Shew that the two expressions $(a^2+1)(b-c)(b+c+1)+(b^2+1)(c-a)(c+a+1)+(c^2+1)(a-b)(a+b+1)$ and $(a^2+1)\times(b-c)+(b^2+1)(c-a)+(c^2+1)(a-b)$ are equal.

127. Find the square root of $(a^2+b^2)(c^2+d^2)+2(ac+bd)\times(bc-ad)$ and shew that $12^{2n}-3^{2n}-2^{4n}+1$ is divisible by 120 (n integer).

128. Simplify: $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ac(x-b)^2}{(b-a)(b-c)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$.

If $\dfrac{a^{2m}+b^{2m}}{a^{2m}-b^{2m}}=\dfrac{(2a-b)^m+b^m}{(2a-b)^m-b^m}$, then $a=b$.

129. Solve (i) $(5-x)^3-8(7-x)^3+(9-x)^3=0$.

(ii) $axy=c(bx+ay)$; $bxy=c(ax-by)$.

(iii) $x+y+z=3$, $ax+by+cz=a+b+c$,

$bcx+acy+abz=ab+bc+ca$.

130. A man buys 570 oranges, some at 16 for a shilling and the rest at 18 for a shilling; he sells them all at 15 for a shilling and gains 3 shillings; how many of each sort does he buy?

XIV.

131. Shew that $(1+xz)(1+yz)^2-\{(1-xz)(1-yz)+2xyz\}^2$ $=4(x+y-xy)(xyz^3+xyz^2+z)$.

132. If $x+y+z=0$, find the H.C.F. of $x^5+y^5+z^5$ and $x^7+y^7+z^7$.

133. Shew that $\dfrac{(1+a)\sqrt{1+b^2}-(1+b)\sqrt{1+a^2}}{a-b}$

$$=\frac{2(1-ab)}{(1+a)\sqrt{1+b^2}+(1+b)\sqrt{1+a^2}}.$$

134. If x^3+ax^2+bx+c be divisible by x^2+px+q, prove that $p(p-a)=q-b$.

If $a+b+c=0$ and $a(by+cz-ax)=b(cz+ax-by)$

$=c(ax+by-cz)$, then $x+y+z=0$.

135. Find the values of a and b in order that $x^2+3xy+4y^2$ may *exactly* divide $x^6+7x^5y+6x^4y^2+6x^3y^3+ax^2y^4+bxy^5$ $+12y^6$.

136. If $x=\dfrac{a+b-c}{a+b+c}$, then $\dfrac{a+bx^2}{b+ax^2}=\dfrac{(a-b+c)^2+4ab}{(b-a+c)^2+4ab}$.

If $(a+b)^2=c(a-b)$, prove that $b(a+b+c)^2(a+b-c)$ $+2abc(a+b+c)-4abc(a+b-c)-ac(a+b+c)(a+b-c)=8abc^2$.

137. Eliminate x, y, z from $ax^2+by^2+cz^2=ax+by+cz$ $=xy+yz+zx=0$.

138. If p and q be the roots of the equation $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}=0$, find the value of $(p-a)(q-a)$.

139. Solve (i) $\dfrac{5}{x+3}+\dfrac{6}{x+4}=2$.

(ii) $\left(\dfrac{x}{a}\right)^{\frac{1}{4}}+\left(\dfrac{x}{b}\right)^{-\frac{1}{4}}=a^{-\frac{1}{4}}+b^{\frac{1}{4}}$.

(iii) $x^{\frac{2}{3}}+a^{\frac{2}{3}}=(x+a)^{\frac{2}{3}}$.

(iv) $x^2+xy=66$; $x^2-y^2=11$.

(v) $cx-by+az=a^2+c^2$; $bx+ay-cz=a^2+b^2$;
$ax+cy+bz=b^2+c^2$.

140. An officer can form his men into a *hollow* square 4 deep and also into another 8 deep. If the front in the latter formation contain 16 men fewer than in the former formation, find the number of men.

XV.

141. Find the square roots of:—

$$x^4+\frac{x^2}{4}+\frac{4}{x^2}-x^3+4x-2 \text{ and } 16+5\sqrt{7}.$$

142. If $x^4+px^3+qx^2+rx+s$ be a *perfect* square, shew that $r^2=p^2s$ and $p^3-4pq+8r=0$.

143. If $\dfrac{(a-b)(b-d)+(b-c)(c-a)+(d-a)(a-c)}{(a-c)(c-d)+(b-d)(d-a)+(d-b)(b-c)}=\dfrac{a-c}{b-d}$, then either $a+d=b+c$ or $(a-b)(c-d)=(b-c)^2$.

144. Find the H.C.F. of $5x^3+38x^2-195x-600$ and $4x^3-15x^2-38x+65$ and the L.C.M. of $4a^3+4a^2b-13ab^2-5b^3$ and $6a^4-5a^3b-13a^2b^2+17ab^3-5b^4$.

145. Solve (i) $(x^2+7x+5)^2-3x^2-21x=19$.

(ii) $\dfrac{3}{5-x}+\dfrac{2}{4-x}=\dfrac{8}{2+x}$.

(iii) $\dfrac{x}{a^2+c^2}+\dfrac{x}{b^2+d^2}=\dfrac{a^2+b^2+c^2+d^2}{(ab-cd)^2+(ad+bc)^2}$.

146. Resolve into its partial fractions $\dfrac{18x+66}{(x+1)(x+5)(x+7)}$.

147. Simplify $\dfrac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}$ and shew that, if $a+b+c+d=0$, then $\frac{1}{3}(a^3+b^3+c^3+d^3)$ $=bcd+acd+abd+abc$.

148. If $\dfrac{a}{x}(b-c)+\dfrac{b}{y}(c-a)+\dfrac{c}{z}(a-b)=0$, shew that $\dfrac{x}{a}(z-y)$ $+\dfrac{y}{b}(x-z)+\dfrac{z}{c}(y-x)=0$.

149. If $\dfrac{1}{x^3}\left(\dfrac{y}{z}+\dfrac{z}{y}\right)=\dfrac{1}{y^3}\left(\dfrac{z}{x}+\dfrac{x}{z}\right)=\dfrac{1}{z^3}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$, then each $=\dfrac{2}{xyz}$.

150. Two persons A and B walk from P to Q and back. A starts 1 hour after B, overtakes him 1 mile from Q, meets him 20 minutes afterwards, and arrives at P when B is $\frac{2}{5}$ of the way back. Find the distance from P to Q and the rates at which they walk.

APPENDIX I.

CALCUTTA UNIVERSITY

ENTRANCE EXAMINATION PAPERS.

1858.

1. Explain the rule for the signs in algebraical multiplication, and multiply $7x^{\frac{1}{2}}-3y^{\frac{1}{4}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}$ by $6x^{\frac{1}{3}}-2y^{\frac{2}{3}}+7x^{\frac{2}{3}}y^{1}$.

2. Find a fraction, such that if 1 be subtracted from its numerator the value shall be $\frac{3}{4}$ and if 6 be added to the denominator the value shall be $\frac{1}{2}$.

3. A and B can do a piece of work in 30 days. A and C in 40 days and B and C in 50 days. All three work together for 10 days. If then two be taken away, how long will each of the others take to finish it?

March 1859.

1. Multiply $1-x+x^2-x^3$ by $1+x+x^2+x^3$,

Divide $(x^{\frac{2}{3}}-a^{\frac{2}{3}})(x^{\frac{2}{3}}+a^{\frac{2}{3}})$ by $x^{\frac{1}{3}}+a^{\frac{1}{3}}$; and

Subtract $bcd^2-(a^2-b^2)bd$ from $(a^2+bc)d^2-(a^2-b^2)bd$.

2 Solve the following equations:–

$$(a) \quad 10(x+\tfrac{1}{2})-23=6x\left(\frac{1}{x}-\frac{1}{3}\right); \quad (b)\ 5+7\sqrt{(\tfrac{3}{5}x-6)}=19.$$

3. I bought 25 yards of cloth for Rs. 223-8-0, for a part I paid Rs 8-8-0 a yard, and for the rest Rs. 9-8-0 a yard; how many yards of each were there?

4. If $m:n::p:q$, prove that $\dfrac{(m-n)(m-p)}{m}=(m+q)-(n+p)$.

5. Divide 39 into two such parts that the greater increased by 6 shall be to the less diminished by 3 as 5 to 2.

6. In a right-angled triangle, the base is 8 and the sum of the hypotenuse and perpendicular is 12, it is required to find them.

7. A person has two horses, and a saddle worth Rs. 75; if the saddle be put on the *first* horse, his value becomes *double* that of the *second*; but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by Rs. 350. What is the value of each?

8. There are three numbers, such that the *sum* of the first and second divided by their *product* is $\frac{1}{2}$; the sum of the second and third divided by their product is $\frac{1}{3}$; and the sum of the first and third divided by their product is $\frac{1}{4}$. Find the numbers.

December 1859.

1. Shew that—

$$\{(ax+by)^2+(ay-bx)^2\} \times \{(ax+by)^2-(ay+bx)^2\} = (a^4-b^4)(x^4-y^4).$$

2. Divide $x^6+2x^3y^3+y^6$ by $(x+y)^2$.

3. Resolve $x^{12}-a^{12}$ into its simplest factors . and simplify—

$$\frac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}} \div \frac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}.$$

4. Find the Greatest Common Measure of—

$$x^3+3x^2-9x+5 \text{ and } x^3-19x+30.$$

5. Solve the equations :—

$$(1)\quad \frac{1}{2}\left\{x-\frac{a}{3}\right\}-\frac{1}{3}\left\{x-\frac{a}{4}\right\}+\frac{1}{4}\left\{x-\frac{a}{5}\right\}=0;$$

$$(2)\quad \frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}.$$

1860.

1. Add together $\dfrac{1}{x+3}+\dfrac{x+1}{x^2-3x+9}-\dfrac{2x^2+x+12}{x^3+27}$.

2. Divide $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{1}{2}}+y$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$; and simplify the expressions

$$\frac{a+c}{(x-a)(b-a)}+\frac{b+c}{(x-b)(a-b)} \text{ and } \frac{a^4-b^4}{a^2-2ab+b^2} \times \frac{a-b}{a(a+b)}.$$

3. Solve the following equations :—

$$\frac{1}{x-1}+\frac{1}{x-4}=\frac{2}{x-3}. \quad \ldots\ldots\ldots \quad (1).$$

$$\frac{2x+11}{x+5}-\frac{9x-9}{3x-4}=\frac{4x+13}{x+3}-\frac{15x-47}{3x-10} \quad \ldots\ldots\ldots \quad (2).$$

4. A person bought a picture at a certain price and paid the same price for the frame; if the frame had cost £1 less and the picture 15*s*. more, the price of the frame would have been only half that of the picture. Find the cost of the picture.

1861.

1. Divide $28x^4+13x^2y^2-xy\ \ +15y^4$ by $4x^2+4xy+3y^2$.

2. Reduce $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}$ to the form $\dfrac{1}{a^4-x^4}$.

3. Multiply $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

4. Solve the following equations:

(1) $6\frac{1}{3}-\dfrac{x-7}{3}=\dfrac{4x-2}{5}$; (2) $4x+3=8x-9$;

(3) $\sqrt{(x+9)}=1+\sqrt{x}$; (4) $\left.\begin{array}{l} x+3y=10 \\ 3x+2y=9 \end{array}\right\}$.

1862.

1. Reduce to its simplest form $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}-\dfrac{x^2+y^2}{x^2-y^2}$.

2. Square $a^{\frac{1}{2}}-b^{\frac{1}{2}}+c^{\frac{1}{2}}$, and divide 1 by $(a+b)^2$ giving three terms of the quotient.

3. Prove that if $a:b::c:d$, then $a-b:a::c-d:c$.

4. Solve the following equations:—

(1) $2x+11=7x-14$; (2) $\sqrt{(x+9)}=1+\sqrt{x}$; (3) $\dfrac{a-b}{x-c}=\dfrac{a+b}{x+2c}$.

5. What fraction is that which if 1 be added to the numerator becomes 1, and if 1 be added to the denominator becomes $\frac{1}{2}$?

1863.

1. Prove that $\dfrac{x+y}{y}-\dfrac{x}{x+y}-\dfrac{x^3-x^2y}{x^2y-y^3}=1$.

2. Divide a^8-x^8 by $a+x$. Multiply $x^{\frac{1}{3}}y+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

3. Solve the following equations:—

(1) $\dfrac{2x}{5}+\dfrac{x-2}{3}=2x-7$; (2) $\sqrt{(3x)}-4=\sqrt{(3x+4)}$;

(3) $2x-\dfrac{y-3}{5}=4$, $3y+\dfrac{x-2}{3}=9$.

4. A post is a fourth of its length in the mud, a third of its length in the water and 10 feet above the water; what is its length?

1864.

1. Add together $x^2-(x-y+z)(x+y-z)$, $y^2-(y-x+z)(y+x-z)$ and $z^2-(z-x+y)(z+x-y)$.

2. Multiply $x+y+z-\sqrt{(xy)}-\sqrt{(yz)}-\sqrt{(xz)}$ by $\sqrt{x}+\sqrt{y}+\sqrt{z}$; and divide $x^8+a^4x^4+a^8$ by x^2-ax+a^2.

3 Simplify the expression $\frac{1}{2}\cdot\frac{1}{x-1}-\frac{x-5}{x^2-7x+10}+\frac{1}{2}\cdot\frac{x-6}{x^2-9x+18}$.

4. Solve the equations :—

(*a*) $\frac{x-1}{3}-\frac{x-9}{2}+\frac{3x-2(x-2)}{7}=4\frac{1}{2}$; (*b*) $5x+11y=146,\ 11x+5y=110$

1865.

1. Divide the continued product of $1+x+y$, $1+x-y$, $1-x+y$ and $x+y-1$ by $1+2xy-x^2-y^2$; and resolve $4(uz-xy)^2-(u^2-x^2-y^2+z^2)^2$ into four factors.

2. Find the Greatest Common Measure of $2x^5-11x^2-9$ and $4x^5+11x^4+81$; and reduce $\frac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$ to its lowest terms.

3. Simplify as much as possible *any one* of the following :—

(1) $\frac{x^3}{(x-y)(x-z)}+\frac{y^3}{(y-x)(y-z)}+\frac{z^3}{(z-x)(z-y)}$;

(2) $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)}$;

(3) $\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)}$.

4. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$.

5. Solve *any two* of the following equations:—

(1) $\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc}$;

(2) $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0$;

(3) $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$;

(4) $\left(\frac{a^2}{x}+b\right)^{\frac{1}{2}}+\left(\frac{a^2}{x}-b\right)^{\frac{1}{2}}=c^{\frac{1}{2}}$.

1866-A.

1. If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$, find the value of

$$\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd-ac}-\sqrt{\left(\frac{b^3}{a^3}-\frac{a^3}{c^3}\right)}.$$

2. Divide $6a^3-17a^2b-7ab^2-5b^3$ by $2a-5b$, and [illegible] by $x-y$ and find the continued product of $ax-1$, $x^2+\frac{1}{a^2}$ and $ax+1$.

3. Find the G.C.M of $x^3+6x^2+11x+6$ and $x^3+9x^2+27x+27$ and the L.C.M. of xy, $x-y$ and y^3-x^2y.

4. Simplify the following .—

(1) $\frac{x+3y}{4(x+y)(x+2y)}+\frac{x+2y}{(x+y)(x+3y)}-\frac{x+y}{4(x+2y)(x+3y)}$;

(2) $\frac{a^2+3a+2}{a^2+2a+1}\times\frac{a^2+5a+4}{a^2+7a+10}$.

5. Extract the cube root of $x^6+6x^5+21x^4+44x^3+63x^2+54x+27$

6. Solve the equations :—

(1) $\frac{x-3}{7}-\frac{\frac{x}{2}-3}{3}=\frac{\frac{x}{6}+2}{2}-\frac{x-6}{3}+\frac{x}{8}$;

(2) $\frac{x-a}{b-a}=\frac{x-b}{a-b}$;   (3) $\frac{px-q}{qx+r}=\frac{px+q}{px-r}$.

7. A is twice as old as B and 4 years older than C. The sum of the ages of A, B and C is 96 years. Find B's age.

1866-B.

1. Find the product of the four factors :—

$$x+y+z,\ x+y-z,\ x+z-y,\ z+y-x.$$

Multiply : $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

Divide : $(x+y+z)(xy+xz+yz)-xyz$ by $x+y$

2. Reduce to its simplest form :—

$$\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(x-z)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$$

Find the Greatest Common Measure of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

3. Extract the square root of $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$ and shew that $(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$.

4. Solve the equations :—

1. $\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3}$; (2) $5x+\frac{\cdot02x+\cdot07}{\cdot03}-\frac{x+2}{9}=9\cdot5$.

1867

1 Reduce to its lowest terms $\frac{x^3-x^2-x+1}{x^4+x^3-x-1}$, and find the Greatest Common Measure of $2x^3+9x^2+4x-15$ and $4x^3+8x^2+3x+20$.

2 Simplify : $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x-y}{x+y}\right)-\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$ or shew that $\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$.

3. Prove either of the identities $(ay-bx)^2+(cx-az)^2+(bz-cy)^2=(x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+by+cz)^2$.

$16s(s-a)(s-b)(s-c)=2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$ where $2s=a+b+c$.

4. Solve either of the equations

(1) $(x+\frac{1}{2})(x-\frac{3}{2})-(x+5)(x-3)+\frac{3}{4}=0$;

(2) $\frac{\sqrt{x}+a}{(\sqrt{x}-b)(\sqrt{x}-c)}+\frac{\sqrt{x}+b}{(\sqrt{x}-a)(\sqrt{x}-c)}+\frac{\sqrt{x}+c}{(\sqrt{x}-a)(\sqrt{x}-b)}=0$.

5. Solve the simultaneous equations :—

(1) $\left.\begin{array}{l} ax+by+c=0 \\ a_1x+b_1y+c_1=0 \end{array}\right\}$; (2) $\left.\begin{array}{l} x+5y-4z=5 \\ 3x-2y+2z=14 \\ -10x+8y+z=6 \end{array}\right\}$.

6. Extract the square root of $x^6+8x^5-2x^3+16x^2-8x+1$ or $a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2$.

1868.

1. Given $a=\sqrt{2}$, $b=\sqrt{3}$, $c=\sqrt{4}$ and $d=0$ find the value of $\sqrt{\{(a^2+b^2+c^2)(b^2+c^2)(b^2+d^2)\}}$, and extract the square root of $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$.

2 Simplify : $\left(\frac{a+b}{a-b}+\frac{a^2+b^2}{a^2-b^2}\right)\div\left(\frac{a-b}{a+b}-\frac{a^3-b^3}{a^3+b^3}\right)$ and shew that $1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{(a+b+c)(a+c-b)(b+c-a)(a+b-c)}{4b^2c^2}$

3. Solve the equations :—

$$(1)\ \frac{4x+3}{9}+\frac{29-7x}{12-5x}=\frac{8x+19}{18}; \qquad (2)\ \frac{8x+4}{\sqrt{(x+5)}}=4\sqrt{(x+5)}.$$

4. Find the Greatest Common Measure of x^3+4x^2-5 and x^3-3x+2, and the Least Common Multiple of $x^5-5x^3+x^2+4x-4$ and $x^4+x^3-6x^2-4x+8$.

5. Solve the simultaneous equations

$$\tfrac{1}{2}x+\tfrac{1}{3}y=12-\tfrac{1}{6}z,\ \tfrac{1}{2}y+\tfrac{1}{3}z-\tfrac{1}{8}x=8,\ \tfrac{1}{2}z+\tfrac{1}{3}x=10.$$

6. There is a number, the sum of whose digits is 5, and if 10 times the digit in the place of tens be added to 4 times the digit in the place of units, the number will be inverted. What is the number?

1869.

1. Divide $x^3+y^3+3xy-1$ by $x+y-1$, and extract the square root of $x^4-3x^3+\frac{11}{12}x^2+2x+\frac{4}{9}$.

2. Resolve all the following expressions into factors, and thence find the Highest Common Measure of x^4+2x^2+1, $x^6+x^4-x^2-1$, and x^4-1; and the Lowest Common Multiple of $6x^2-x-1$, $3x^2+7x+2$, and $2x^2+3x-2$.

3. Simplify: $(a)\ \dfrac{x}{x-a}-\dfrac{x}{x+a}-\dfrac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$;

$$(b)\ \frac{a^2+ac}{a^2c-c^3}-\frac{a-c}{(a+c)c}-\frac{2c}{a^2-c^2} \qquad (c)\ \frac{3x^3-2x^2-1}{4x^3-2x^2-3x+1}.$$

4. Solve the equations :—

$$(a)\ \tfrac{1}{8}(x-2)-\tfrac{1}{7}(x-4)=\tfrac{1}{12}(2x-3)-2\tfrac{1}{4}; \quad (b)\ \frac{a}{x}+\frac{b}{y}=m,\ \frac{b}{x}+\frac{a}{y}=n.$$

5. A labourer is engaged for 30 days, on condition that he receives 2s. 6d. for each day he works, and loses 1s. for each day he is idle; he receives £2-7-0 in all. How many days does he work, and how many days is he idle?

1870.

1. Find the product of $3a+2b$ and $3a+2c-3b$, and test the result by making $a=1$, $b=c=3$.

Divide $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4-x^3y+x^2y^2-xy^3+y^4$

and $\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}$ by $\dfrac{a}{b}+\dfrac{b}{a}$.

2. Prove that $\dfrac{1}{1+\dfrac{1}{a\div c}}+\dfrac{1}{1-\dfrac{1}{a\div c}}+\dfrac{2}{1+\dfrac{1}{a^2\div c^2}}=\dfrac{4a^4}{a^4-c^4}$; and show that the notation $\dfrac{\dfrac{a}{b}}{c}$ is of ambiguous meaning.

Simplify the expressions :—

$$\frac{x^{a+b}\,x^{a-b}\,x^{c-2a}}{x^{c}\,x^{a}};\quad \frac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^2}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}.$$

3. Solve the equations :—

(1) $\dfrac{x-3}{5}-\dfrac{x-5}{4}=\dfrac{2}{3}$. (2) $\dfrac{a-x}{a}+\dfrac{2a-x}{2a}=\dfrac{3a-x}{3a}$.

(3) $\dfrac{1}{5x}+\dfrac{y}{9}=5,\ \dfrac{1}{3x}+\dfrac{y}{2}=14.$

4. Find the Least Common Multiple of—

$$x^3+x^2y+xy^2+y^3 \text{ and } x^3-x^2y+xy^2-y^3.$$

5. Extract the square root of

$$4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4.$$

Reduce $\dfrac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}$ to its lowest terms.

6. AB is a railway 220 miles long; and three trains (P, Q, R) travel upon it at rates of 25, 20 and 30 miles per hour respectively: P and Q leave A at 7 A.M. and 8-15 A.M. respectively, and R leaves B at 10-30 A.M. When and where will P be equidistant from Q and R?

1871.

1. Multiply $x^3-\frac{1}{2}x^2y-3y$ by $2x^2-\frac{1}{4}y^2$, and find the square root of $x^4-2x^2-\dfrac{2}{x^2}+\dfrac{1}{x^4}+3$.

2. Reduce $\dfrac{10x^3+19x^2-9}{25x^2-19x+6}$ to the lowest terms, and find the Least Common Multiple of $2(x-2)^2$, $2x^2-8$, x^3+2x, $2x^2-4x$.

3. Simplify :-

(i) $1+\dfrac{a}{b}-\dfrac{b}{a+b}-\dfrac{a^2}{ab-b^2}+\dfrac{2a^2}{a^2-b^2}$;

(ii) $\left(1-\frac{1}{1+x}\right)\left(x+\frac{1}{2+x}\right)\times\frac{x^2-\frac{1}{x}}{1+\frac{1}{x}}\div\left(1+x+\frac{1}{x}\right)$.

4. Solve the equations :—

(i) $\frac{6}{3x-5}-\frac{1}{x-5}=\frac{2}{2x-5}$; (ii) $\left.\begin{array}{l}4x-\frac{1}{3}(2y-3)=6\frac{1}{3}\\ 3y-\frac{2}{3}(3x-1)=7\end{array}\right\}$

5. A and B compared their monthly incomes and found that A's income was to that of B as 7 to 9, and that the third of A's income was Rs. 30 greater than the difference of their incomes. Find what each received.

1872.

1. Divide x^4-10x^2+9 by x^2-2x-3, and find the G.C.M. of $3x^3-17x^2+19x+11$ and $6x^3-25x^2+17x-22$.

2. Simplify —

(i) $\left\{\frac{2a}{x^2-a^2}-\frac{1}{x-a}+\frac{2}{x+a}\right\}\times\frac{x^2}{x-a+\frac{a^2}{x}}$;

(ii) $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}+\frac{1}{z(z-x)(z-y)}$.

3. Solve the equations :

(i) $\frac{7x-1}{4}-\frac{1}{3}\left(2x-\frac{1-x}{2}\right)=6\frac{1}{3}$; (ii) $\left.\begin{array}{l}\frac{x-y}{3}=\frac{y-1}{4}\\ \frac{4x-5y}{7}=x-7\end{array}\right\}$.

4. Rs. 1,100 are so divided among A, B and C, that if A were to give B Rs. 200, B would then have twice as much as A, and three times as much as C. How many rupees did A, B and C each receive originally?

5. If $a:b::c:d$, prove that $a:b::a\pm c:b\pm d$; also shew that $a^2+c^2:b^2+d^2::(a+c)^2:(b+d)^2$.

1873.

1. Reduce to their simplest forms :—

(i) $\frac{3x}{2}\sqrt{\frac{400y^2}{81x^2}}$; (ii) $\frac{(x^3-y^3)(x+y)^2}{(x^2+xy+y^2)(x^2-y^2)}$;

(iii) $\frac{2+x}{2(x+1)}+\frac{2-x}{2(x-1)}+\frac{x}{x^2+1}$.

2. Find the G.C.M. of $x^4-9a^2x^2+10a^3x$ and $ax^3-a^2x^2-4a^4$; and the L.C.M. of $3ax^2-3a^2x$, x^2-a^2, a^2+ax, $\sqrt{(3ax)}$, $\sqrt{x}-\sqrt{a}$.

3. Solve the equations:—

$$(a)\ \frac{12}{x+2}=6-2\left(\frac{3x+2}{x+1}\right);\qquad (b)\ \sqrt{x}-\sqrt{(4+x)}=\frac{2}{\sqrt{x}};$$

$$(c)\ 2x-\frac{2y-1}{3}=3\tfrac{1}{2}+\frac{3x-2y}{4},\ 4y-\frac{5-2x}{4}=6-\frac{3-2y}{5}.$$

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each of these ratios $=\frac{a+c+e}{b+d+f}$.

Assuming that $\frac{a+b-c}{a+b}=\frac{b+c-a}{b+c}=\frac{c+a-b}{c+a}$, and that $a+b+c$ is *not* $=0$, shew that $a=b=c$.

5. Two persons started at the same time from A. One rode on horseback at the rate of $7\frac{1}{2}$ miles an hour and arrived at B 30 minutes later than the other, who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between A and B.

1874.

1. Simplify:—

$$(i)\ \frac{x^{m+2n}y^{3m-8n}}{x^{5m}y^{6n}};\qquad (ii)\ \frac{a}{a+b}-\frac{a+b}{2b}+\frac{a^2+b^2}{2b(a-b)};$$

$$(iii)\ \frac{\frac{a^3}{b^3}-\frac{b^3}{a^3}}{\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}-1\right)}\times\frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{ab}}.$$

2. Find the Least Common Multiple of $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$.

Extract the square root of $9x^4-2x^3y+\frac{163}{9}x^2y^2-2xy^3+9y^4$.

3. Solve the equation $\frac{15-\frac{2}{3}x}{5}-\frac{2x+5}{2\frac{1}{2}}=\frac{17-\frac{1}{3}x}{3}$.

The expression $ax-3b$ is equal to 30 when x is 3, and to 42 when x is 7; what is its value when x is $4\dot{3}$; and for what value of x is it zero?

4. Shew that if $a:b::c:d$, (i) $a+b:a-b::c+d:c-d$;

$$(ii)\ 4(a+b)(c+d)=bd\left(\frac{a+b}{b}+\frac{c+d}{2}\right)^2.$$

5. A certain number consists of two digits; the left-hand digit is double the right-hand digit, and if the digits be inverted; the ratio of the number thus formed to 60 is 4 : 5. Find the number.

1875.

1. Subtract $3a-\frac{2}{3}b+\frac{3}{4}c$ from $2a+\frac{1}{2}b-\frac{1}{4}c$; multiply $\frac{x^2}{y^2}+1+\frac{y^2}{x^2}$ by $\frac{x}{y}-\frac{y}{x}$ and find the G.C.M. of $6x^4+x^3-6x^2-5x-2$ and $2x^4+3x^3+2x^2-7x-6$.

2. Simplify: $\frac{1+x+x^2}{1-x}+\frac{1-x+x^2}{1+x}-\left(\frac{x}{1+x}+\frac{1-x}{x}-\frac{1+x}{x}\right)\frac{1}{1-x}$, and shew that if $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c+a}{b}=1$ and $a-b+c$ is *not* $=0$, then $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$.

3. Solve the equations:

(i) $\frac{x-4}{5}-\frac{7x-3}{6}+\frac{3}{7}=0$ (ii) $\sqrt{5x-1}=1+\sqrt{5x-2}$;

(iii) $4x-\frac{1}{3}(5y-4)=1$, $\frac{3y-2x}{4}+\frac{1}{3}x=\frac{1}{2}$.

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each of the fractions is equal to

(i) $\frac{ka+lc+me}{kb+ld+mf}$, (ii) $\left(\frac{ace}{bdf}\right)^{\frac{1}{3}}$.

5. How many bundles of hay at Rs. 5 per thousand must a *ghas wala* mix with 5,600 bundles at Rs. 6 per thousand in order that he may gain 20 per cent. by selling the whole at 11 annas per hundred?

1876.

1. Simplify the following expressions:—

(1) $3a-[a+b-2\{a+b+c-(a-b+c-d)\}+a]$.

(2) $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y)$;

(3) $$\frac{\frac{a}{a-b}-\frac{a}{a+b}}{\frac{b}{a-b}-\frac{b}{a+b}}\div\frac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}\times\frac{a^2}{a^2+b^2}$$

2. Find the Greatest Common Measure of $2x^3+3x^2+2x-2$ and $4x^4-2x^3+2x-1$. Multiply x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$.

Find the square root of $4-4c+2b+c^2-bc+\frac{b^2}{4}$.

3. Solve the equations :—

(1) $\frac{2x-13}{9}-\frac{x-1}{11}=\frac{x}{8}+\frac{x}{7}-9$; (2) $\frac{x+y}{2}+\frac{3x-5y}{4}=2, \frac{x}{14}+\frac{y}{18}=1.$

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

(1) $\frac{ma+nb}{mc+nd}=\frac{b^2c}{d^2a}$; (2) $a^2+c^2+e^2 : b^2+d^2+f^2 :: ce : df.$

5. A can do a piece of work in 9 days, B in twice that time, C can only do ½ as much as A in a day; how long would A, B, and C, working together, require to do the same piece of work ?

1877.

1. Simplify ; $\frac{x+2}{1+x+x^2}-\frac{x-2}{1-x+x^2}-\frac{2x^2-4}{1-x^2+x^4}$; multiply together $a+b+c$, $b+c-a$, $c+a-b$, $a+b-c$; and divide $x^4+x^3-24x^2-35x+57$ by x^2+2x-3.

2. Solve the equations :—

(1) $\frac{2x-3}{6}+\frac{3x-8}{11}=\frac{4x+15}{33}+\frac{1}{2}$;

(2) $2(x+2)=1+\sqrt{(4x^2+9x+14)}$;

(3) $3x+4y-11=0$, $5y-6z+8=0$, $7z-8x-13=0$.

3. Find the Greatest Common Measure of $x^4+x^3-11x^2-9x+18$ and $x^4-10x^3+35x^2-50x+24$.

4. Find the first four terms of the square root of a^2+x^2, and from the result deduce the square root of 101 correct to six places of decimals.

5. If $a : b=c : d$, prove that $a^2+c^2 : b^2+d^2 :: ac : bd$.

6. A and B together can do a piece of work in 15 days. A can do it alone in 24 days ; how long would B take to do it alone ?

7. Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5*s*. 2*d*. and 9*s*. 10*d*. respectively ; but if the luggage had all belonged to one of them, he would have been charged 19*s*. 2*d*. How much luggage is each passenger allowed to carry free of charge, and how much luggage had each passenger ?

1878.

1. Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$ by $1+xy+yz+zx$.

2. Extract the square root of-

$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(bz-cy)^2-(cx-az)^2-(ay-bx)^2.$$

3. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, find the value of $(b-c)x+(c-a)y+(a-b)z$.

4. Solve the equations :—

(a) $\sqrt{4x^2+20x+17}-\sqrt{16x^2+11x+10}=2(x+2)$;

(b) $\frac{4x+3}{9}+\frac{13x}{108}=\frac{8x+19}{18}$.

5. Simplify the expression $(4x^3-3x)^2\left\{\frac{3\frac{\sqrt{1-x^2}}{x}-\frac{(1-x^2)^{\frac{3}{2}}}{x^3}}{1-3\left(\frac{1-x^2}{x^2}\right)}\right\}^2$.

1879.

1. Multiply $a^{2n}-a^nx^n+x^{2n}$ by a^n+x^n, and find the Greatest Common Measure of $x^2+\frac{7}{6}x+\frac{1}{3}$ and $x^2+\frac{2}{3}x+\frac{1}{12}$.

2. Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$; and Simplify $\frac{x-y}{x-z}+\frac{x-z}{x-y}-\frac{(y-z)^2}{(x-y)(x-z)}$.

3. Solve the equations :—

(a) $x-k+\sqrt{k^2+x^2}=m$;

(b) $\begin{cases} a^x\, a^{y+1}=a^7 \\ a^{2y}\, a^{x+3}=a^{2} \end{cases}$; (c) $\begin{cases} \frac{4}{x}+\frac{10}{y}=2 \\ \frac{3}{x}+\frac{2}{y}=\frac{19}{20}. \end{cases}$

4. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$.

Two armies number 11,000 and 7,000 men respectively; before they fight, each is reinforced by 1,000 men: in favour of which army is the increase?

5. From two towns 561 miles apart two men start, one from each, at the same time · one goes 24 and the other 27 miles a day: in how many days will they meet?

1880.

1. Simplify $\left\{\frac{x}{a}+\frac{2x^2}{a(b-x)}\right\}\left\{\frac{a}{x}-\frac{2ax}{x(b+x)}\right\}$.

2. Find the Highest Common Factor and Least Common Multiple of $3x^2-10ax+7a^2$ and $x^3-5ax^2+7a^2x-3a^3$.

3. Solve the equations :

(a) $15+\sqrt{(x+7)}=19$; (b) $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$;

(c) $\frac{7+x}{5}-\frac{2x-y}{4}=3y-5$; $\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x$.

4. If $a:b::c:d$, shew that

$$ma+nc : mb+nd :: (a^2+c^2)^{\frac{1}{2}} : (b^2+d^2)^{\frac{1}{2}}.$$

5. Extract the square root of

$$x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+a^{\frac{8}{5}}.$$

6. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours: it also goes up-stream 40 miles and down-stream 55 miles in 13 hours. Find the rate of the stream and of the boat.

1881.

1. What do you mean by a negative quantity?
Prove that $a-(b-c)=a-b+c$.

2. Simplify $\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}+\frac{1}{b(b-a)(x-b)}$ and resolve into elementary factors the expressions :—

$$x^2-5ax-66a^2 \text{ and } (1-c^2)(1+a)^2-(1-a^2)(1+c)^2.$$

3. A man receives $\frac{x}{y}$ths of Rs. 10 and afterwards $\frac{y}{x}$ths of Rs. 10. He then gives away Rs. 20. Shew that he cannot lose by the transaction.

4. What is an *equation*? Prove that a simple equation has only one root.

5. Solve the equations :—

(1) $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3$;

(2) $\frac{4\cdot05}{9x}-\frac{\cdot3}{\cdot8-2x}=\frac{1\cdot8}{x}-\frac{3\cdot6}{2\cdot4-6x}$; (3) $ax+by=c$, $a^2x+b^2y=c^2$.

6. *A* challenged *B* to ride a bicycle race of 1,040 yards. He first gave *B* 120 yards' start, but lost by 5 seconds: he then gave *B* 5 seconds' start and won by 120 feet. How long does each take to ride the distance?

1882.

1. Divide x^n-a^n by $x-a$, and find the continued product of $x-a$, x^2+ax+a^2, x^3+a^3.

2. Resolve into factors $x^2+13x+12$, x^2+x-12, $343x^3+512y^3$.

3. Find the Highest Common Factor of $x^3-7x^2-80x+576$ and $3x^2-14x-80$, and the Lowest Common Multiple of these two expressions and $3x^2+17x-90$.

4. If $a:b::c:d::e:f$, shew that each of these ratios is equal to $\sqrt[3]{a^3+c^3+e^3} : \sqrt[3]{b^3+d^3+f^3}$.

5. Solve the equations :—(i) $(6x+9)^2+(8x-7)^2=(10x+3)^2-71$.

(ii) $\cdot65x+\frac{\cdot585x-\cdot975}{6}=\frac{1\cdot56}{\cdot2}-\frac{\cdot39x-\cdot78}{\cdot9}$;

(iii) $\frac{x-2}{2}-\frac{x+y}{14}=\frac{x-y-1}{8}-\frac{y+12}{4}$.

$\frac{x+7}{3}+\frac{y-5}{10}-1=-\frac{5(y+1)}{7}$.

6. The distance from a place P to another place Q is $3\frac{1}{2}$ miles : two persons A and B start together from P to go to Q, the former by carriage which travels at the rate of 6 miles an hour, the latter walking at the rate of 3 miles an hour If A remains at Q for 15 minutes and then returns by the carriage to P, find where he will meet B.

1883.

1. Divide—

$(a-b)^2c+(a-b)c-(c-a^3)b^2+(c-a)b$ by $(a-b)c^3-(c-a)b^2$.

2 Find the Greatest Common Measure of—

$x(6x^2-8y^2)-y(3x^2-4y^2)$ and $2xy(2y-x)+4x^3-2y^3$.

3. Simplify : $\frac{2}{a+x}-\frac{1}{a-x}+\frac{3x}{a^2-x^2}+\frac{ax}{a^2+x^2}$.

4. Find the value of $\frac{x^2-y^2+x}{y^2-x^2+y}$, when $x=\frac{a-b}{a+b}$ and $y=\frac{a+b}{a-b}$.

5. Solve the equations :—(a) $x^2+y^2=a^2$, $xy=b^2$;

(b) $\frac{x^2-2\frac{1}{2}}{4}-\frac{x-3\frac{1}{2}}{5}=\frac{2x^2-3}{8}-\frac{x-5\frac{1}{2}}{3}$; (c) $\frac{x}{\cdot5}+\frac{1}{\cdot05}+\frac{x}{\cdot005}-\frac{1}{\cdot0005}=0$;

(d) $3y+x-2=0$, $3z-4y=x+15$, $2x+7z=7$.

6. Reverse the digits of a number and it will become five-sixths of what it was before ; also the difference between the two digits is one.

2. Divide—

$(ax+by)^3+(ax-by)^3-(ay-bx)^3+(ay+bx)^3$ by $(a+b)^2x^2-3ab(x^2-y^2)$.

3. Extract the square root of (1) $(ab+ac+bc)^2-4abc(a+c)$;

(2) $x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2+2(y^3+y^2z+yz^2+z^3)x$
$+y^4+2y^2z^2+z^4$.

4. Solve the following equations :—

(1) $\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8}{3}\cdot\frac{x-\frac{1}{2}}{x-2}$; (2) $\frac{x+2\frac{1}{2}}{15}+\frac{x+3\frac{1}{3}}{25}=\frac{x+4\frac{1}{6}}{55}$;

(3) $(x+7)(y-3)+7=(y+3)(x-1)+5$, $5x-11y+35=0$.

5. The dimensions of a rectangular court are such that if the length were increased by 3 yards, and the breadth diminished by the same, its area would be diminished by 18 square yards, and if its length were increased by 3 yards, and its breadth increased by the same, its area would be increased by 60 square yards; find its dimensions.

6. Find the G.C.M. of—

$4x^3-8ax^2-20a^2x+24a^3$ and $6x^3+24ax^2+6a^2x-36a^3$.

7. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{(a+c)^3}{(b+d)^3}=\frac{a(a-c)^2}{b(b-d)^2}$.

1889.

1. Solve the equation : $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{1}{3}$.

2. Find the square root of $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$.

3. If $\frac{x}{(b-c)(b+c-2a)}=\frac{y}{(c-a)(c+a-2b)}=\frac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$.

4. Reduce to its lowest terms $\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$.

5. A man rides one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour. If he had travelled at a uniform rate of $3c$ miles per hour he could have ridden from A to B and back again in the same time.

Prove that $\frac{2}{c}=\frac{1}{a}+\frac{1}{b}$.

6. Simplify the expression :—

$(16x^5-20x^3+5x)^2+(1-x^2)\{16(1-x^2)^2-20(1-x^2)+5\}^2$.

1890.

1. Multiply $ax^{\frac{1}{4}}+3a^{\frac{1}{2}}x^{\frac{1}{2}}+4x^{\frac{3}{4}}$ by $a-3a^{\frac{1}{2}}x^{\frac{1}{4}}+4x^{\frac{1}{2}}$, and find the Greatest Common Measure of x^5+x^4+1 and $x^6-2x^4+x^2-1$.

2. Extract the square root of $9x^2-24x+19-\frac{4}{x}+\frac{1}{4x^2}$

3. Solve the equations :—(i) $\frac{1}{3}(x+1)+\frac{1}{2}(x-1)-\frac{1}{4}(3x-7)=2$;

(ii) $\frac{1}{x-a}-\frac{1}{x-a+c}=\frac{1}{x-b-c}-\frac{1}{x-b}$; (iii) $\frac{1}{3x}-\frac{1}{7y}=\frac{2}{3}$; $\frac{1}{2x}-\frac{1}{3y}=\frac{1}{6}$.

4. Of the candidates in a certain examination 45 per cent. passed. If there had been 30 more candidates of whom 19 failed, the number of successful candidates would have been 41·8 per cent. How many candidates were there ?

5. If $a:b::c:d$, shew that (i) $a-b : a+b :: c-d : c+d$, and (ii) $a^2d-bc^2=ac(b-d)$.

1891.

1. Divide $x+6a^{\frac{1}{5}}x^{\frac{4}{5}}+6a^{\frac{3}{5}}x^{\frac{2}{5}}+a+5a^{\frac{2}{5}}x^{\frac{3}{5}}+7a^{\frac{4}{5}}x^{\frac{1}{5}}$ by $x^{\frac{1}{5}}+a^{\frac{1}{5}}$.

2. Solve the following equations :—

$$(a)\quad x-\frac{3-x}{5}=3.\quad \frac{x-1}{2}+\frac{x+1}{5}-\frac{3}{10}.$$

$$(b)\quad \frac{1}{(a-b)(x-a)}-\frac{1}{(c-d)(x-c)}=\frac{1}{(a-b)(x-b)}-\frac{1}{(c-d)(x-d)}.$$

3. A tradesman sells two articles together for 46 rupees, making 10 per cent. profit on one and 20 per cent. on the other. If he had sold each article at 15 per cent. profit, the result would have been the same. At what price does he sell each article ?

4. Prove the rule for finding the Greatest Common Measure of two numbers, a and b.

Find the G.C.M. of $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$......, each of these ratios will be

$$\left\{\frac{pa^n+qc^n+re^n+\ldots\ldots}{pb^n+qd^n+rf^n+\ldots\ldots}\right\}^{\frac{1}{n}}.$$

If $a+b : b+c = c+d : d+a$, prove that $a=c$ or $a+b+c+d=0$.

1892.

1. Find the value of :—

$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right] \text{when } x=-\tfrac{1}{2} \text{ and } y=2$$

2. Find the square root of $\dfrac{4a^2-12ab-6bc+4ac+9b^2+c^2}{4a^2+9c^2-12ac}$

3. Solve the equations :—

(a) $\frac{1}{5}(x-8)+\frac{4+x}{4}+\frac{x-1}{7}=7-\frac{23-x}{5}$; (b) $\frac{x+a}{x+b}=\frac{x+3a}{x+a+b}$.

4. The express leaves Bristol at 3 P.M. and reaches London at 6 P.M.: the ordinary train leaves London at 1-30 P.M., and arrives at Bristol at 6 P.M. If both trains travel uniformly, find the time when they will meet

5. If $x : y = y : z$, find the simplest value of: $\dfrac{xyz(x+y+z)}{(xy+yz+zx)}$.

1893.

1. Find the H.C.F. of: $3x^3-5x^2+5x-2$ and $2x^4-2x^3+3x^2-x+1$.

2. Extract the square root of: $4x^6-12x^5+13x^4-22x^3+25x^2-8x+16$

3. Solve the equations :—

(1) $120x-4[5x-2\{6x+7(x-8)\}]$.

$$=16-4\left[3x-2\left\{x-6(x-1)\right\}\right]:$$

(2) $\frac{x+b}{a-b}=\frac{x-b}{a+b}$; (3) $\frac{6}{x}+\frac{4}{y}=3$; $\frac{9}{x}-\frac{1}{y}=2\frac{3}{4}$.

4. Divide the number 834 into 2 parts such that 30 per cent. of one part exceeds 40 per cent. of the other part by 6.

5. If $a : b = c : d$, prove that $ma-nb : a+b = mc-nd : c+d$.

What number must be added to each of the numbers 3, 5, 7, 10 to give four numbers in proportion?

APPENDIX II.

BOMBAY UNIVERSITY.

MATRICULATION EXAMINATION PAPERS.

1859.

1. If $a=2$, $b=3$, $c=4$; find the value of—

$$\frac{a-b+c}{a+b-c}+\frac{b-c+a}{b+c-a}+\frac{c-a+b}{c+a-b}.$$

2. Add together $5xy-3az+7$, $6xy+4bz-3$, $4az-2xy+c$ and $-19+az-xy$.

3. Prove that $a-(c-d)=a-c+d$.

4. Multiply $a^{\frac{5}{2}}+2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}+8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}+32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-2b^{\frac{1}{3}}$.

5. Divide $x^{-1}-y^{-1}$ by $x^{-\frac{1}{3}}-y^{-\frac{1}{3}}$.

6. Are you acquainted with any rule which will enable you to know that (x^3-3x^2+3x-1) is divisible by $(x-1)$ without a remainder? If so, state the rule generally.

7. Find the value of $(a+c-b)^3+(a+b-c)^3+(b+c-a)^3+24abc$.

8. Shew that $\dfrac{x}{x+y}+\dfrac{y}{x-y}=\dfrac{x}{x-y}-\dfrac{y}{x+y}$.

9. Solve the following equations:—

(i) $\frac{2}{3}x+\frac{1}{4}x=5$. (ii) $\dfrac{3x-13}{7}+\dfrac{11-4x}{3}=0$.

(iii) $4(x^2-1)+\frac{2}{3}(x^2-2x+1)=6(x-1)+3x-3$.

10. A man, whose age is 40, has a son who is 9 years old. When will the father be twice as old as his son?

11. Suppose in solving a problem as the last, you were to obtain $x=-$(quantity), how would you explain the result? Illustrate your answer by stating problem 10 in a different manner.

12. A farmer sells to one person 9 horses and 7 cows for Rs. 3,000 and to another 6 horses and 13 cows at the same prices and for the same sum; what was the price of each?

1860.

1. State what is meant by a *term* of an expression. Distinguish between *like* and *unlike* quantities.

If $a=4$, $b=3$, $c=2$, find the value of $\dfrac{2a+b(2^2-a)}{3b-\sqrt{(2c^3)}}$.

2. Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$ and a^m-2c^n by $a-c^n$.

3. Divide a^4-81 by $a-3$ and $\left(\dfrac{3}{\sqrt{5}}-\dfrac{\sqrt{5}}{3}\right)^5\sqrt{x^3}$ by $4\sqrt{\dfrac{3}{5x}}$.

4. Extract the square root of $\frac{2}{3}x^3+2+\dfrac{8}{3x}+\dfrac{4x^2}{9}+\dfrac{4}{x^4}+\dfrac{x^4}{4}$.

State and prove the rule which you employ.

5. Prove the rule for finding the Greatest Common Measure of two or more quantities.

6. Find the G.C.M. of $6a^2+7ax-3x^2$ and $6a^2+11ax+3x^2$; and deduce therefrom that of $72a^3x+84a^2x^2-36ax^3$ and $24a^2x+44ax^2+12x^3$.

7. Reduce the following fractions to their simplest forms:—

$$\frac{7x-10}{5}-\frac{3x-7}{6}-\frac{27x-30}{30};\quad 2\frac{x^2-\frac{1}{4}}{2x+1}+\tfrac{1}{2};$$

$$\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)} \text{ and}$$

$$\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}+\frac{1}{x^n-1}+\frac{1}{x^n+1}.$$

8. Solve the equations:—

$$\cdot15x+\frac{\cdot135x-\cdot225}{6}=\frac{\cdot36}{\cdot2}-\frac{\cdot09x-\cdot18}{\cdot9};\quad \frac{7x-4}{x-1}=\frac{7x-26}{x-3}.$$

9. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100 minutes. Had the stream been half as strong again, they would have taken $31\frac{1}{4}$ minutes longer. Find the rate of the stream.

1861.

1. Remove the brackets from the expression:—

$a-[5b-\{a-(3e-3b)+2e-(a-2b-e)\}]$ and collect the result into its simplest form.

2. Write down the rule for the multiplication of algebraical expressions and multiply $x^5+a^5-ax(x^3+a^3)$ by $x^3+a^3-ax(x+a)$.

3. Define the G.C.M. of two or more quantities, and write down clearly the algebraical method by which it is found.

Find the G.C.M. of—

$a^4-pa^3-p^2a^2-p^3a-2p^4$ and $3a^3-7pa^2+3p^2a-2p^3$.

4. Simplify $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}} \times \dfrac{m^2-n^2}{m^3+n^3}$.

5. Solve the following equations:—(1) $\dfrac{7x+2}{5}-\dfrac{4x-1}{2}=0$.

(2) $2b\{\sqrt{(x+a)}-b\}+2c\{\sqrt{(x-a)}+c\}=a$. (3) $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

6. The distance between two termini A and B, is 100 miles. A train starting from A runs up-hill during the first 30 miles, the next 50 are on a level, and the remaining 20 up-hill. The train travels 5 miles an hour faster on the level than when ascending the hill. There are stoppages at C, D, E, and F, at distances of 20, $42\frac{1}{2}$, $67\frac{1}{2}$ and 90 miles respectively from A, each of which occupies 3 minutes. Find the time of the arrival at C, D, E, and F of the train which starts from A at 8 o'clock and arrives at B at 42 minutes past 12.

1862.

1. (*a*) Define the *power* of a number, and the *index* of the power; and illustrate the distinction between them by any numerical example.

(*b*) Give the origin of the expressions:—

The *square* and *cube* of a number. If these names are not perfectly correct, substitute ones which are; and give your reasons.

(*c*) Find the value of a^0.

2. Define a *Simple quantity* and a *Compound quantity*. Is $42abx^2$ a simple or a compound quantity? Give the *names* of the different kinds of compound quantities, illustrated by *algebraical examples*.

3. Multiply $x^2-\frac{1}{2}x+\frac{1}{4}$ by $\frac{1}{2}x+2$. Divide (1) x^4+6x^3-5 by x^2+3.

(2) $\dfrac{a+b}{a+c}$ by $\dfrac{a-c}{a-b}$. (3) $1-\dfrac{y^2}{y^2+a^2}$ by $1+\dfrac{a^2}{y^2+a^2}$.

4. The epitaph of Diophantus states that he passed the sixth part of his life in childhood, and the twelfth part of it in the state of youth, and that after an interval of five years more than the seventh part of his life, he had a son who died when he had attained to half the age of his father

and that the father survived the son four years. How long did Diophantus live?

5. To the sum of $\frac{a}{y}$ and $\frac{y}{x}$ annex -2 and multiply the rusult by xy, putting the product in its simplest form.

6. Find two numbers, the greater of which shall be to the less as their sum is to 42, and as their difference to 6.

7. Reduce $\dfrac{a^4-8a^2+a-6}{a^3+6a^2+10a+3}$ to its lowest terms.

8. Resolve into elementary factors :-

(1) $25a^2x^2-4y^2$. (2) x^3+y^3. (3) x^3-y^3.

9. Find (a) the instant of time between 3 and 4 o'clock at which the hour-hand and the minute-hand are exactly in *the same direction* and (b) that at which they are exactly *opposite* each other.

10. Solve the following simultaneous equations :—

$$4x-5y+6z=3,\ 8x+7y-3z=2,\ 7x+8y+9z=1.$$

1863.

1. Explain the following :—*Like quantities, and expression of n dimensions, a co-efficient, a root of any quantity.*

If $a=4$ and $x=2$, find the numerical value of—

$$\frac{2ax^2}{(a-x)^2}-\frac{6\sqrt[3]{(ax)}}{a^3\sqrt{(2a+4x)}}-\frac{29x^2}{64a}.$$

2. State the rule for the addition of algebraical quantities.

Add together $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-\frac{yz}{a}-\frac{xz}{b}-\frac{xy}{c}$, $ax+by+cz+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

3. Divide—

$a^2x^2-b^2x^2+a^2bx-ab^2x+a^3-a^2b+ab^2-b^3$ by $ax^2+bx^2+abx+a^2+b^2$.

4. Investigate a rule for finding the L.C.M. of two algebraical expressions. Find the L.C.M. of a^2-1, a^3-1 and a^2+5a+4.

5. Define a fraction. Reduce to their simplest forms—

$$\left(\frac{a^2}{x}-\frac{x^3}{a^2}\right)\div\left(\frac{a}{x}+\frac{x}{a}\right),\quad \frac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}-\frac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}.$$

6. Shew that the sum of any fraction and its reciprocal is always greater than 2.

7. Interpret the meaning of the equation $\frac{a^m}{a^n}=a^{m-n}$ in the case of m and n being equal.

Extract the square root of $a^3-2a^{\frac{3}{2}}b^{\frac{2}{3}}+2a^{\frac{3}{2}}c^{\frac{3}{2}}-2b^{\frac{2}{3}}c^{\frac{3}{2}}+b^{\frac{4}{3}}+c^3$.

8. Solve the following equations :—

(*a*) $6-\frac{x-1}{2}-\frac{x-2}{3}=\frac{3-x}{4}$, (*b*) $\frac{5}{x}+\frac{4}{y}=58$, $\frac{3}{x}+\frac{7}{y}=67$.

(*c*) $\frac{\sqrt{(x+1)}-\sqrt{(x-1)}}{\sqrt{(x+1)}+\sqrt{(x-1)}}=\frac{1}{2}$.

9. A vessel can be filled by means of one tap in 3 hours and by means of a second tap in 5 hours, in what time will it be filled if both taps run together ?

10. A ship left Bombay on a voyage of 3 weeks, with provisions for that time at the rate of a seer a day for each man. At the end of a week a storm arose which washed 4 men overboard and so damaged the vessel that its speed was reduced by half, and each man could be allowed only $\frac{3}{4}$ of seer *per diem*. What was the original number of the crew ?

1864.

1. In what respect are the sciences of arithmetic and algebra identical, and in what respect do they differ ?

Explain the following :—*Simple quantities, Similar quantities, Irrational quantities.*

2. If $a=1$, $b=2$, $c=3$, $d=4$, find the numerical value of the expression,

$$\frac{ab}{bc}-\frac{bc}{cd}+\frac{ac}{da}-\frac{cd}{da}+\frac{ad}{dc}+\frac{da}{ab}.$$

3. From $(a+b)x+(a+c)y$ take $(a-b)x-(b-c)y$.

4. What is the value of $a\times 0$? Multiply $a^4+a^3b+a^2b^2+ab^3+b^4$ by $(a-b)$.

5. Find the G.C.M. of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

6. Prove that the value of a fraction is not altered by multiplying its numerator and denominator by the same quantity.

Reduce to their simplest forms :—

(i) $\frac{a-b}{ab}-\frac{a-c}{ac}+\frac{b-c}{bc}$. (ii) $\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b^2+ax)}$.

7. If $x+\frac{1}{c}=p$, prove that $x^3+\frac{1}{x^3}=p^3-3p$.

8. Solve (1) $\frac{\sqrt{(x+2)}}{\sqrt{(x-2)}}=2$. (2) $\frac{m}{x}+\frac{n}{y}=a,\ \frac{n}{x}+\frac{m}{y}=b.$

9. Find a number such that whether divided into two equal parts, or into three equal parts, the product of the parts shall be the same.

10. A labourer is engaged for 10 days, on condition that he shall receive 8 annas for every day's work done and that he shall pay 1 anna for every day on which he is absent from work—at the end of the 10 days he receives 8 annas; on how many days was he absent?

1865.

1. Define the terms *Homogeneous expressions*, *Similar quantities*.

Remove the brackets from the expression—

$$a-[2b+\{3c-3a-(a+b)\}+2a-(b-3c)].$$

2. If $x=2$, $y=3$, $a=6$, $b=5$, find the values of—

(i) $\{\sqrt[3]{b(x+y)^2}\}+\sqrt[3]{\{(x+a)(b-2x)\}}+\sqrt[3]{\{x(b-y)^2\}}$.

(ii) $x+2a-\{y+b-[x-a-(y-2b)]\}$.

3. State the rule for the multiplication of two algebraical quantities. Multiply together $(a^2-3a+2)^2$ and a^2+6a+1.

4. Assuming the rule for finding the G.C.M. of two algebraical quantities, prove that for finding the G.C.M. of three such quantities. Find by inspection the G.C.M. of :—

$$(x-1)^2(x-2)(x-3) \text{ and } (x-1)^3(x-4)(x-5).$$

5. Simplify the expression—

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}.$$

And multiply $\frac{x^2}{a^2}-\frac{x}{a}+1$ by $\frac{x^2}{a^2}+\frac{x}{a}+1$.

6. Extract to three terms the square root of $1+x$.

7. Divide $a^{\frac{5}{2}}-a^{\frac{3}{2}}b+ab^{\frac{3}{2}}-2a^{\frac{1}{2}}b^2+b^{\frac{5}{2}}$ by $a^{\frac{3}{2}}-ab^{\frac{1}{2}}+a^{\frac{1}{2}}b-b^{\frac{3}{2}}$.

Shew that $\frac{a\sqrt{(a+x)}}{\sqrt{(a+x)}-\sqrt{x}}=a+x+\sqrt{(ax+x^2)}$.

8. Solve (i) $\frac{2x-3}{3x-4}=\frac{4x-5}{6x-7}$. (ii) $\sqrt{(x+48)}+\sqrt{x}=12$.

(iii) $\sqrt[m]{(a+x)}=\sqrt[2m]{(x^2+8ax+b^2)}$.

9. A cistern can be filled by two pipes, *A* and *B*, in 12 hours, and by the pipe *A* alone in 20 hours; required the time in which it would be filled by *B* alone.

10. A privateer sailing at the rate of 10 miles an hour discovers a ship 18 miles off running from her at the rate of 8 miles an hour, how many miles can the ship run before being overtaken?

1866.

1. If $a=16$, $b=10$, $x=5$, $y=1$, find the numerical value of—

$$(a-y)\{\sqrt{(2bx)}+x^2\}+\sqrt{\{(a-x)(b+y)\}}.$$

2. If the multiplicand and the multiplier of any expression be both homogeneous, what will be the character of the product?

3. Divide $6a^4-a^3b+2a^2b^2+13ab^3+4b^4$ by $2a^2-3ab+4b^2$.

4. Prove that—

$$\{(a-b)^2+(b-c)^2+(c-a)^2\}^2=2\{(a-b)^4+(b-c)^4+(c-a)^4\}.$$

5. Find the G.C.M. of $4x^4+9x^3+2x^2-2x-4$ and $3x^3+5x^2-x+2$.

6. What is a complex fraction? How do you simplify it?

Simplify $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

7. Extract the square roots of—

(i) $4x^4-4x^3+5x^2-2x+1$. (ii) $x^4-x^3+\dfrac{x^2}{4}-4x-2+\dfrac{4}{x^2}$.

8. Shew that $a^0=1$, $a^{-1}=\dfrac{1}{a}$ and $(a^{\frac{3}{2}})^{\frac{1}{2}}=a^{\frac{3}{4}}$.

9. Solve (i) $\dfrac{2x-3}{3x-4}=\dfrac{4x-5}{6x-7}$. (ii) $\dfrac{3x-5y}{2}+3=\dfrac{2x+4}{5}$, $8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3}$.

10. A pound of tea and three pounds of sugar cost six shillings, but if sugar were to rise 50 per cent., and tea 10 per cent., they would cost seven shillings. Find the price of tea and sugar.

11. A railway train after travelling for one hour meets with an accident, which delays it one hour, after which it proceeds at $\frac{3}{5}$ths of its former rate, and arrives at the terminus three hours behind time; had the accident happened 50 miles further on, the train could have arrived 1 hour 20 minutes sooner. Required the length of the line.

1867.

1. If $a=3$, $b=-2$, $c=1$, find the values of—

(i) $\dfrac{a^2+b^2+c^2}{a+b+c}-\dfrac{a^2-b^2-c^2}{a-b-c}$.

(ii) $a(a-b)(a-c)\ \{\sqrt{(2a^2+3b+4c)}-\sqrt{(a+b)}\sqrt{(a+c)}\}$.

2. Multiply $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

3. Extract the square root of $x^6-2x^4y^2+2x^3y^3+x^2y^4-2xy^5+y^6$.

4. Shew that $\dfrac{x+y}{\sqrt{(x+y)}-y}=\dfrac{\sqrt{(x^3+x^2y-xy^2-y^3)}}{\sqrt{(x^2-y^2)}-y\sqrt{(x-y)}}$.

5. Reduce (i) $\dfrac{3}{4(x^2-y^2)}+\dfrac{3}{x-y}-\dfrac{41}{x+y}$. (ii) $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+3}{2-x}}}$.

6. Divide $x^{2-a-b}\times x^{b-\frac{3}{2}}$ by $x^{-(a-\frac{1}{3})}$ representing the result in its simplest form.

7. Add together $\dfrac{1}{a+b}$, $\dfrac{1}{a-b}$, and $\dfrac{2b}{a^2-b^2}$.

8. Solve (i) $\sqrt{(x+12)}+\sqrt{x}=6$. (iii) $\dfrac{2x-3}{5x-2}=\dfrac{2x+6}{5x+37}$.

(iii) $\dfrac{5}{y}+\dfrac{2}{x}=\dfrac{5}{6}$, $\dfrac{4}{y}+\dfrac{3}{x}=\dfrac{9}{10}$

9. Find the exact time after 3 o'clock that the hour and minute hands are 1st exactly in the same direction, and, 2ndly at right angles to each other.

10. A merchant has a certain number of Back Bay and Mazagon Shares. The market rate for the two Shares was Rs. 2,000, but Mazagon Shares rose 10 per cent. and Back Bay fell 20 per cent. The value of the two shares became in consequence $12\frac{1}{2}$ per cent. less than before. Find the original market value of each share.

11. Two boats start at the same time from Bassein and Tanna, the distance between which is 18 miles. At a distance of one mile from Tanna the Callian creek falls into the Tanna creek, causing a current at the rate of $2\frac{1}{2}$ miles an hour towards Tanna, and two miles an hour towards Bassein. The boat from Tanna is rowed at the rate of $3\frac{1}{2}$ miles per hour, and the Bassein boat at 3 miles per hour. Where will they meet?

1868.

1. Substitute $y+3$ for x in $x^4-x^3+2x^2-3$ and arrange the result.

2. Find the Greatest Common Measure of x^5-y^5 and x^2-y^2.

3. Find the least Common Multiple of x^2-4a^2, $x^3+2ax^2+4a^2x+8a^3$, and $x^3-2ax^2+4a^2x-8a^3$.

4. If the denominators have some factors in common, how do you proceed in reducing fractions to a common denominator?

Reduce the following: $\frac{5}{2(x+1)}-\frac{2}{10(x-1)}-\frac{24}{5(2x+3)}$

5. Simplify the following:—

(i) $\left(\frac{a}{a+b}+\frac{b}{a-b}\right)\div\left(\frac{a}{a-b}-\frac{b}{a+b}\right)$; (ii) $\frac{a^4-b^4}{a^2-2ab+b^2}+\frac{a-b}{a^2+ab}$.

6. Solve the following:—

(i) $\frac{x}{4}-\frac{5x+8}{6}=\frac{2x-9}{3}$. (ii) $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$.

7. Two shepherds owning flock of sheep agree to divide its value, A takes 72 sheep, and B takes 92 sheep and pays A Rs. 350. Required the value of a sheep.

8. Solve the following:—

(i) $ax+by=c$; $mx-ny=d$. (ii) $\frac{x}{a}+\frac{y}{b}=1$; $\frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}$.

9. A and B start together from Poona to go to Kirkee. A would reach Kirkee half an hour before B, but missing his way goes a mile and back again needlessly, during which he walks at twice his former rate, and reaches Kirkee 6 minutes before B. C starts 20 minutes after A and B, and walking at the rate of $2\frac{1}{2}$ miles per hour arrives at Kirkee ten minutes after B. Find the rates of walking of A and B and the distance from Poona to Kirkee.

10. Extract the square roots of—

(1) $4x^4-12ax^3+25a^2x^2-24a^3x+16a^4$.

(2) $4a-12a^{\frac{1}{2}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}+16a^{\frac{1}{2}}c^{\frac{1}{4}}-24b^{\frac{1}{3}}c^{\frac{1}{4}}+16c^{\frac{1}{2}}$.

11. What is a surd quantity? Can a quadratic surd be made up of two others which have not the same irrational part?

Extract the square root of $4+2\sqrt{3}$.

1869.

1. Define a *term* and an *expression*.

What is meant by the *dimension* or the *degree* of a term?

Write down two trinomial homogeneous expressions, one of six dimensions, and the other of seven dimensions.

2. Reduce to its simplest form—

$$\{2x^2-(y^2-xy)\} - \{y^2-(4x^2-y^2)\} + \{2y^2-(3xy-x^2)\}.$$

3. Divide $a^5-2a^4b+2a^3b^2-4a^2b^3-8ab^4+16b^5$ by a^2-2b^2.

4. Resolve $a^2-4b^2-c^2+9d^2+2(3ad-2bc)$ into factors.

5. Find the G.C.M. of $4a^5x+6a^4x^2-18a^3x^3$ and $12a^5x^2-34a^4x^3+28a^3x^4-6a^2x^5$.

6. Find the value of : $\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{2a^2}{a^2-b^2}\right)\dfrac{a-b}{2a}$.

7. Extract the square root of $x^6-4x^5y+4x^4y^2-2x^3y^2+4x^2y^3+y^4$.

8. Solve (i) $\dfrac{4x-9}{27}-\dfrac{x-3}{4}=\dfrac{5x-3}{6}-\dfrac{x+6}{2}$.

(ii) $\dfrac{75-x}{3(x+1)}+\dfrac{80x+21}{5(3x+2)}=\dfrac{23}{x+1}+5$.

9. The charge for the first class tickets of admission to an exhibition was Rs. 4 each and the charge for second class tickets was Rs. 2-8-0. The whole number of tickets sold was 259, and the total amount received for them was Rs. 731-8-0. How many first class tickets were sold, and how many second class tickets?

10. Given $\dfrac{9x-2y}{30}-\dfrac{2x-2y}{36}=\frac{1}{5}$ and $\dfrac{5x-6y}{60}+1\frac{1}{10}=2x-2\frac{1}{3}$, find the values of x and y.

11. *A* and *B* (one of whom could do the work alone in a less number of days than the other) agree to reap a field for Rs. 20. If they had worked together every day, the field would have been reaped in 15 days; but at the end of 7 days, *A* left off working for four days; and it consequently took $16\frac{1}{2}$ days to reap the field. In how many days could *A* alone, and in

how many days could B alone, have reaped the field; and what part of the Rs. 20 ought each to receive for the work he actually did?

1870.

1. Find the value of —

$$\frac{ab^2c+(a^2d-b^3)c-bd^2}{(bc^2-ad^2)^3} \text{ when } a=2,\ b=3,\ c=4,\ d=5.$$

2. Define a *power* of a quantity, and the *index* of the power.

Interpret a^{-m} and $\frac{a^m}{a^n}$.

3. Multiply $a^{(m-1)n}-b^{(n-1)m}$ by a^n-b^m.

4. Find the L.C.M. of—

$2x^4+2x^3y-18x^2y^2+22xy^3-8y^4$ and $3x^4-3x^3y-9x^2y^2+15xy^3-6y^4$.

5. Reduce $\left(\frac{x^2}{yz}+\frac{x}{z}-\frac{x+z}{w}\right)\div\left(\frac{x}{y}-\frac{z}{w}\right)$.

6. If $s=\frac{1}{2}(a+b+c)$, prove that $s(s-a)(s-b)+s(s-b)(s-c)+s(s-c)\times(s-a)=(s-a)(s-b)(s-c)+abc$.

7. Extract the square root of $x+6x^{\frac{5}{6}}y^{\frac{1}{3}}+9x^{\frac{2}{3}}y^{\frac{2}{3}}-4x^{\frac{1}{2}}y^{\frac{1}{2}}-12x^{\frac{1}{3}}y^{\frac{5}{6}}+4y$.

8. Solve (*a*) $\frac{3x+1}{26}-\frac{402+x}{12}=99-\frac{371-6x}{2}$.

(β) $\frac{10x+4}{21}+\frac{7-2x^2}{14(x-1)}=\frac{11-5x}{15}+\frac{4x-3\frac{2}{3}}{6}$.

(γ) $10x-\frac{4y+3}{4}=7+\frac{12y-10x}{5}$; $16y+\frac{5x-2}{3}=26\frac{1}{2}-\frac{8y+1}{2}$.

9. A garrison of 1,500 men was victualled for 36 days; but after 16 days it was reinforced, and the provisions were then exhausted in 12 days. Required the number of men in the reinforcement.

10. A person left Poona in the Sattara mail buggy at 2 P.M. and having proceeded a certain distance he went out of the buggy and returned to Poona on foot, walking at the rate of 3 miles an hour, and he reached Poona at 8 P.M. Had he gone 6 miles further in the buggy he would not have got back to Poona till 10 hours 40 minutes P.M. How far did he go towards Sattara and what was the speed of the buggy?

1871.

1. Required the product of $\sqrt[2]{(a)}$ and $\sqrt[4]{(a^2)}$.

2. Find the value of $\dfrac{x^4-a^4}{x^2-2ax+a^2} \div \dfrac{x^2+ax}{x-a}$.

3. Find the value of—

$$\frac{1}{4a^3(a-x)}+\frac{1}{4a^3(a+x)}+\frac{1}{2a^2(a^2+x^2)}.$$

4. Find the G.C.M. of $6x^3+13x^2+6x$ and $8x^3+6x-9x^2$.

5. Extract the square root of $a^4-2a^3+\frac{3}{2}a^2-\frac{1}{2}a+\frac{1}{16}$.

6 Reduce to its lowest terms the following fraction :

$$\frac{x^4+7x^3+7x^2-15x}{x^3-2x^2-13x+110}.$$

7. Solve the equation $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{2x}{5}$.

8. Solve the equations—

(α) $u+2x+y+2z=-3$ (β) $2u+x+2y+z=3$.

(γ) $u+2x+12y+z=21$. (π) $u+x+6y+z=10$.

9. A farmer parting with his stock sells to one person 9 horses and 7 cows for Rs. 3,000 and to another at the same prices 6 horses and 13 cows for the same sum. What was the price of each.

10. A garrison had sufficient provisions for 30 months, but at the end of 4 months the number of troops was doubled, and 3 months after, it was reinforced with 400 men more, on which account the provisions lasted only 15 months altogether. Required the number of men in the garrison before the augmentation took place.

1872.

1. Explain the meaning of the words *expression*, *term*, *factor*, *coefficient*, *power*, *index*. Write down an expression of two terms, each of two factors ; another of two factors, each of three terms, each of four factors ; another of four factors, each of three terms, each of two factors.

2. State and prove *the rule of signs* in multiplication and division.

3. Divide the product of $y^3-12y+16$ and $y^3-12y-16$ by y^2-16.

4. Divide $(x^2-yz)^3+8y^3z^3$ by x^2+yz.

5. Resolve $x^{16}-y^{16}$ into 5 factors.

6. State and prove the rule for finding the G.C.M. of two algebraical quantities. Find the G.C.M. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

7. Divide $a^2+1+\frac{1}{a^2}$ by $\frac{1}{a}-1+a$ and $\frac{x^2}{2y^2}-4+\frac{6y^2}{x^2}$ by $\frac{x}{2y}-\frac{3y}{x}$.

8. Solve the equations :—

(1) $\frac{2x-1}{3}-\frac{3x-2}{4}=\frac{5x-4}{4}-\frac{7x+6}{12}$. (2) $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

(3) $(x+\frac{3}{2})(x-\frac{5}{2})=(x+3)(x-5)+\frac{9}{4}$.

(4) $(x-\frac{3}{2})(x+\frac{5}{2})=(x-3)(x+5)+1$.

9. Divide Rs. 15,200 among Krishna, Gopal and Govind, so that Gopal shall have Rs. 1,000 more than Krishna, and Govind Rs. 2,700 more than Gopal.

10. A certain sum is to be divided among Rama, Lakshman and Hanuman; Rama is to have Rs. 300 less than half, Lakshman is to have Rs. 100 less than the third part and Hanuman Rs. 80 more than a quarter. How much will each receive?

11. Gopal and Govind agree to divide their flock; Gopal takes 72 sheep; Govind takes 92 sheep, and pays Gopal Rs. 300. Required the value of a sheep.

12. One-tenth of a rod is coloured red; one-twentieth orange; one-thirtieth yellow; one-fortieth green; one-fiftieth blue; one-sixtieth indigo; and the remainder, which is 302 inches long, violet. Find the length of the rod.

1873.

1. Explain the terms—*index, co-efficient, like and unlike quantities.*

2. If $a=\sqrt{2}$; $b=\sqrt{3}$; $c=\sqrt{4}$; $d=0$; find the value of

$$\sqrt{\{(a^2+b^2+c^2)(b^2+c^2)(b^2+d^2)\}}\ .$$

3. Divide $a^{2n+1}-a^{n+1}-a^n+a^{n-1}$ by a^{n-1}.

4. Simplify—(i) $\frac{(x-1)^2}{y^3}\times\frac{(x+1)y^2}{x-1}$. (ii) $\frac{a}{(1-a)^2}-\frac{a^2}{(1-a)^3}+\frac{1}{(1-a)}$.

5. Find the G.C.M. of x^3+2x^2+2x+1 and x^3-2x-1 and the L.C.M. of $6(x^2+xy)$; $8(xy-y^2)$ and $10(x^2-y^2)$.

6. Extract the square root of $4x^4+12x^3y+13x^2y^2+6y^3+y^4$.

7. Solve: (i) $\frac{1}{3}(x+1)+\frac{1}{4}(x+3)=\frac{1}{5}(x+4)+16$.

(ii) $\frac{x+1}{2}+3y=33$; $\frac{y+6}{3}+2x=27$.

8. The sum of two numbers is 100 and the greater is to the less as $7:3$; what are the numbers?

9. There is a number consisting of two digits whose sum is 10, and if 72 be subtracted from it, the digits will be inverted. What is the number?

1874.

1. Explain the terms—*binomial*, *exponent*, *rational* and *irrational quantities*.

2. Find the values of the following:—

(i) $\dfrac{x^4-y^4}{a^2b+ab^2}\times\dfrac{a+b}{(x+y)^2}\div\dfrac{(x-y)^2}{ab}$. (ii) $\dfrac{3x+2}{x-1}-\dfrac{x-1}{4x+1}+2x$.

3. Find the G.C.M. of $12a^2x^4+120a^4x^2-132a^5x$, and $3x^8a^2-27x^7a^3+39x^6a^4-15x^3a^7$; and the L.C.M. of $7(x-a)$, $14(x^2-a^2)$, and $21(x^3-a^3)$.

4. Find the square root of $x^6-4x^5+10x^4-20x^3+25x^2-24x+16$, and the cube root of $8x^3+60x^2y+150xy^2+125y^3$.

5. Solve the equation:—

$$\frac{4x^2+2xy+288-6y^2}{2x+13-2y}=2x+3y-131;\ 5x-4y=22.$$

6. B has 5 miles start of A, but only travels at the rate of 3 miles per hour while A travels at the rate of $4\frac{1}{2}$ miles per hour. Where will A overtake B, and how long will he take to do it?

7. A mixture is made up of a gallons at m rupees per gallon, b gallons at n rupees and c gallons at p rupees per gallon; what will be the value of the mixture?

8. Solve the equations:—

$$\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=-\frac{33}{10};\ \frac{1}{x}+\frac{1}{y}-\frac{1}{z}=-\frac{37}{60};\ \frac{5}{x}-\frac{2}{y}+\frac{3}{z}=-\frac{191}{60}$$

9. The sum of the three digits of which a number consists is 9; the first digit is one-eighth of the number consisting of the last two; and the last digit is likewise one-eighth of the number consisting of the first two. Find the number.

1875.

1. Describe in words the operations indicated by the expression:—

$$\sqrt{\left\{\frac{(a^3+b^3)c^2}{(a^2-b^2)^3}\right\}}.$$

2. Multiply $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

Divide $x^{\frac{7}{6}}-2x^{\frac{5}{6}}+x^{\frac{1}{2}}$ by $x^{\frac{2}{3}}-2x^{\frac{1}{2}}+x^{\frac{1}{3}}$.

3. Find the G.C.M. of $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-ab-3b^2}$ and reduce the fraction to its lowest terms. Find the L.C.M. of a^3+x^3 and a^2-x^2.

4. Divide $\dfrac{a}{1+a}+\dfrac{1-a}{a}$ by $\dfrac{a}{1+a}-\dfrac{1-a}{a}$.

5. Extract the square root of $a^2-2ax+x^2+2a-2x+1$.

6. Solve the equations:—(i) $(3x-1)^2+(4x-2)^2=(5x-3)^2$.

$$\text{(ii)}\quad \frac{3x-1}{\sqrt{(3x-1)}}=1+\frac{\sqrt{(3x)}-2}{2}.$$

(iii) $x+y-z=3$; $x+z-y=5$; $y+z-x=7$.

7. A person has a number of rupees which he tries to arrange in the form of a square. On the first attempt he has 116 over. When he increases the side of the square by three rupees he wants 25 to complete the square. How many rupees has he?

1876.

1. Find the value of (i) $\dfrac{\sqrt{(3-2x^2)}-x}{x(1+3x)-x^3}$, when $x=-\frac{1}{3}$.

$$\text{(ii)}\quad 1-[1-\{1-(1-\overline{1-1})\}].$$

What is the use of *brackets?* What is the rule for removing brackets which are preceded by a *minus* sign?

2. Simplify the expressions:—

$$\text{(i)}\quad \frac{9a^2b^2}{16(x+y)}\div\left\{\left(\frac{3a(x-y)}{7(c+d)}\div\frac{4(c-d)}{21ab^2}\div\frac{c^2-d^2}{4(x^2-y^2)}\right)\right\}.$$

$$\text{(ii)}\quad \left\{\sqrt{\left(\frac{a+x}{x}\right)}-\sqrt{\left(\frac{x}{a+x}\right)}\right\}^2-\left\{\sqrt{\left(\frac{x}{a}\right)}-\sqrt{\left(\frac{a}{x}\right)}\right\}^2$$

$$+\frac{x^2}{a(a+x)}.$$

3. What is meant by a *co-efficient?* Find the co-efficient of x in the quotient obtained by dividing $8x^4+xy^3-y^4$ by $x-\dfrac{y}{2}$.

4. Separate into their simplest factors:—

(i) $x^2-xy-6y^2$. (ii) $x^3-4xy^2-x^2y+4y^3$.

Find the highest common divisor of—

$$1+x^{\frac{1}{2}}+x+x^{\frac{3}{2}} \text{ and } 2x+2x^{\frac{3}{2}}+3x^2+3x^{\frac{5}{2}}.$$

5. If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$,

Shew that $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right)=0.$

6. State and prove the rule for finding the L.C.M. of two or more algebraical expressions.

Find by inspection the L.C.M. of $2x^3-8$; $3x^3-9x+6$ and $6x^2+18x+12$.

7. Extract the square root of $4x^4-12x+25x^{-2}-24x^{-5}+16x^{-8}$ and the cube root of $8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27$.

8. Distinguish between an *equation* and an *identity*, and give an example of each. What value of c makes $(x-2)^2-(x-1)(x-3)=c$ an identity? Can any value of c make it an equation?

9. If the telegraph posts by the side of a railway be 60 yards apart, shew that twice the number passed by a train in a minute gives roughly the number of miles per hour at which the train is moving. If 11 posts be passed in a minute, in what time would the distance traversed, estimated by this rule, be 1 mile in error?

10. A boy receives a fixed sum as pocket-money at the beginning of every week, and in each week he spends half of all that he had at its beginning. He had no money before the first pocket money was given him and at the end of the third week he has 1s. 2*d*. What was his weekly allowance?

1877.

1. Remove the brackets from the expression:—

$$\{m-n-\overline{3x-2y}\}-\{2x+5y-\overline{n+m}\}$$

and enclose the last three terms of the expression $a-b+c-2d-1$ in a bracket with a negative sign.

2. Find the quotient which arises from dividing the third power of $10a^2$ by the square root of one million times a^{12}.

3. Extract the square root of $x^4-6x^3y+13x^2y^2-12xy^3+4y^4$.

4. Find the G.C.M. of $21x^3-26x^2+8x$ and $6x^2-x-2$ and the L.C.M. of x^2-1; x^2+2x-3 and x^3-7x^2+6x.

5. Reduce (i) $a+\left(\frac{b-a}{1+ba}\right)\times\frac{a}{b}\div\left(1-a\frac{b-a}{1+ba}\right)$ and (ii) $\frac{3\frac{1}{4}-\frac{1}{3}(x-2)}{1\frac{1}{12}+(x-\frac{3}{2})}$.

6. Solve (i) $(x-a)(x-b)=ab-x^2$.

(ii) $\sqrt{(x+4)}+\sqrt{(2x+9)}=\sqrt{(3x+25)}$.

(iii) $\frac{4}{x}-\frac{5}{y}=\frac{x+y}{xy}+1\frac{22}{25}$; $xy=\frac{35}{34}(y-x)$.

7. From a certain sum of money I took away one-third part and put in its stead Rs. 50; from the sum thus increased I took away one-fourth part and put in its stead Rs. 70. I then found I had Rs. 120; what was the original sum?

8. A certain number consists of two digits whose sum is 8; another number is obtained by reversing the digits. If the product of these two is 1855; find the number.

1879-80

1. Reduce to the simplest form

$a^2+2d^2-(2e^2-b^2)-\{(d^2-e^2-c^2)+(d^2-c^2)\}$.

2. Find the square root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{9}{4}$.

3. Find the G.C.M. of $2x^2-xy-6y^2$ and $3x^2-8xy+4y^2$.

4. Add together $\frac{x(x+3)}{(x+1)(x+2)}$ and $\frac{2}{3x(x+2)}$ and find the value of the result when $x=\frac{1}{2}$.

5. Find the values of x and y from the equations—

$$ax+by=c^2;\quad \frac{a}{b+y}-\frac{c}{a+x}=0.$$

6. A and B invest equal sums in speculation; A gains Rs. 1,000 and B loses so much that his money is now $\frac{2}{3}$ of A's money. If each gave the other $\frac{1}{4}$ of his present sum, B's loss would be diminished by one half. What did each adventure?

1880-81.

1. Find the continued product of the following quantities:

$a+b+c$; $-a+b+c$; $a-b+c$; $a+b-c$; $a+b\sqrt{(-1)}$; $a-b\sqrt{(-1)}$.

2. From the sum of the squares of $\frac{1}{a-b}$ and $\frac{1}{a+b}$ subtract the square of $\frac{2b}{a^2-b^2}$.

3. Find the G.C.M. of the quantities forming the numerator and denominator of the fraction $\frac{x^4-15x^2+28x-12}{2x^3-15x+14}$ and reduce the fraction to its lowest terms.

4. Find the values of x, y and z in the following set of simultaneous equations:—

$x+y+z=6$; $3x-y+2z=7$; $4x+3y-z=7$.

5. Determine the *time*, between ten and twelve o'clock, at which the hour and minute hands of a common clock are exactly together.

6. One student said to another: "If you give me half your money I shall have a hundred rupees." The other replied: "I shall have hundred rupees if you give me a third of your money." How much had each?

1881-82.

1. Divide $a^2x^8+(2ac-b^2)x^4+c^2$ by ax^4+c-bx^3.

2. Resolve $4a^2b^2-(a^2+b^2-c^2)^2$ into four factors.

3. Find the square root of $\frac{x^2}{y^2}+\frac{y^2}{4x^2}-\frac{x}{y}+\frac{y}{2x}-\frac{3}{4}$.

4. Find the G.C.M. and L.C.M. of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

5. A train carrying three classes of passengers, at 6 as., 4 as. and 3 as., has eight times as many third class passengers as there are of the second class, and seven times as many second class passengers as there are of the first class. The whole sum received is Rs. 290-6-0. How many first class tickets were issued?

6. Solve the following equations:—(1) $\frac{2x-9}{27}+\frac{x}{18}-\frac{x-3}{4}=8\frac{1}{3}-x$.

(2) $1{\cdot}2x-\frac{{\cdot}18x-{\cdot}05}{{\cdot}5}={\cdot}4x+8{\cdot}9$. (3) $\frac{x+6}{y}=\frac{3}{4}$; $\frac{x}{y-2}=\frac{1}{2}$.

1882-83.

1. (1) If $\left(a+\frac{1}{a}\right)^2=3$, prove that $a^3+\frac{1}{a^3}=0$.

(2) If $x+y=2z$, shew that $\frac{x}{x-z}+\frac{y}{y-z}=2$.

2. Resolve into their simplest factors the following expressions; $6x^2+5x-6$; $3x^2-10x-8$; $9x^4-82x^2y^2-9y^4$.

3. Find the G.C.M of $x^4-3x+20$, $5x^4-3x^3+64$.

4. Extract the cube root of:—

$$x^3+\frac{8}{x^3}-12x^2-\frac{48}{x^2}+54+\frac{108}{x}-112.$$

5. Solve the equation:—

$$\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}.$$

6. A father's age is four times that of his eldest son and five times that of his younger son; when the elder son has lived to three times his present age, the father's age will exceed twice that of his younger son by three years. Find their present ages.

7. A cottage costs Rs. 1,500 to build. At what rent must it be let to pay five per cent. clear after allowing ten per cent. of the receipts for repairs?

1883-84.

1. Shew that—

$$\frac{x}{a}+\left(\frac{z-x}{b}\right) \text{ and } \frac{z^2}{(a^2+b^2)}+\frac{a^2+b^2}{a^2b^2}\left(x-\frac{za^2}{a^2+b^2}\right)^2 \text{ are identical}$$

expressions, such that the one can be deduced from the other.

2. Divide $x^3+8y^3-27z^3+18xyz$ by $x-3z+2y$ and resolve into factors of the first degree $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

3. A criminal having escaped from prison, travelled 10 hours before his escape was known. He was pursued so as to be gained upon 3 miles an hour. After his pursuers had travelled 3 hours, they met an express going at the same rate as themselves, who met the criminal 2 hours 24 minutes before. In what time after the commencement of the pursuit will they overtake him?

4. Divide the number 127, into four such parts, that the first increased by 18, the second diminished by 5, the third multiplied by 6, and the fourth divided by $2\frac{1}{2}$, shall all be equal.

5. Solve the equation $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$.

And find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$ when $x=\frac{4ab}{a+b}$.

1884-85.

1. Divide $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$.

2. If $(a+b)(b+c)(c+d)(d+a)=(a+b+c+d)(bcd+cda+dab+abc)$, then prove that $ac=bd$.

3. Shew that—

$2(a-b)(a-c)+2(b-c)(b-a)+2(c-b)(c-a)$ is the sum of three squares, and resolve $10x^2-23x-5$ into the simplest factors.

4. Find the Greatest Common Measure of $7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$ and $8x^4-13ax^3+5a^2x^2-3a^3x+3a^4$.

5. A certain number consisting of two digits becomes 110 when the number obtained by reversing the digits is added to it; also the first number exceeds unity by five times the excess of the second number over unity. What is the number?

6. A person walks from A to B, a distance of $7\frac{1}{2}$ miles, in 2 hours $17\frac{1}{2}$ minutes and returns in 2 hours 20 minutes, his rates of walking uphill, down-hill and on a level road being 3, $3\frac{1}{2}$ and $3\frac{1}{4}$ miles per hour respectively. Find the length of the level road between A and B.

1885-86.

1. Find the continued product of—

x^2-2y^2 ; $x^2-2xy+2y^2$; x^2+2y^2 ; and $x^2+2xy+2y^2$.

2. Explain the terms *factor*, *multiple* and *common measure*. Resolve into factors, and thence find the L.C.M. of

$a^2+6ab+5b^2$; $a^3-a^2b-ab^2+b^3$; and $a^2+5ab-6b^2$.

3. Simplify the following fractional expressions :—

$$\left(x+\frac{16x-27}{x^2-16}\right)\div\left(x-1+\frac{13}{x+4}\right);\text{ and }\frac{\sqrt{(a^4-4a^3+12a^2-16a+16)}}{\sqrt[3]{(a^9+24a^6+192a^3+512)}}.$$

4. Solve the equations—

(i) $\dfrac{2x}{3}-\dfrac{1-\frac{1}{2}x}{4x}=\dfrac{2x+7}{12}-\dfrac{1-x}{2}$. (ii) $\dfrac{p}{x}+\dfrac{q}{y}=0$; $px+qy=r$.

5. I bought a horse and a carriage for £90; I sold the horse at a gain of 12 per cent. and the carriage at a loss of 4 per cent., and gained on the whole 6 per cent. Find the prime cost of the carriage.

6. A man walks one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour, and travelling back from B to A at the rate of $3c$ miles per hour takes the same time. Prove that $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}$.

1886-87.

1. (i) Simplify $24\,\{x-\frac{1}{2}(x-1)\}\,\{x-\frac{2}{3}(x-2)\}\,\{x-\frac{3}{4}(x-1\frac{1}{3})\}$; and subtract the result from $\dfrac{(x^2+7x+12)(x^2-x-6)}{x-3}$.

(ii) Reduce to its simplest form—

$$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$$

2. Resolve into factors $4\,(ad-bc)^2-(a^2+d^2-b^2-c^2)^2$.

If $x+\dfrac{1}{x}=p$, express $x^3+\dfrac{1}{x^3}$ in terms of p.

3. Find the G.C.M. and L.C.M. of

$x^5+x^4-4x^3+2x^2+6x-9$; x^4-x^2+6x-9 ; and $x^4+2x^3-5x^2-6x+9$.

4. Extract the square root of $\dfrac{x^3}{16}-\dfrac{x^2}{6}-\dfrac{x^{\frac{3}{2}}}{4}+\dfrac{x}{9}+\dfrac{x^{\frac{1}{2}}}{3}+\dfrac{1}{4}$ and the cube root of $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$.

5. Solve the equations—

(i) $\sqrt{\{(x-a)^2+2ab+b^2\}}=x-a+b$. (ii) $\frac{15}{x}-\frac{1}{y}=4\frac{1}{2}$; $\frac{9}{x}+\frac{2}{y}=4$.

6. A man walks from the University towards Malabar Hill at the rate of 3 miles an hour, runs part of the way back at the rate of $8\frac{1}{4}$ miles an hour, and then walks the remainder in 1 hour 5 minutes. He was out 2 hours 44 minutes: find how far he had gone.

1887-88.

1. If $a+b=c+d$, prove that either of them is equal to

$\frac{abcd}{ab+cd}\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}$; and if $x+\frac{1}{y}=1$ and $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$.

2. Simplify—

(i) $$\frac{(b+c)(x^2+a^2)}{(c-a)(a-b)}+\frac{(c+a)(x^2+b^2)}{(a-b)(b-c)}+\frac{(a+b)(x^2+c^2)}{(b-c)(c-a)}.$$

(ii) $$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)-\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{y}\right).$$

3. Shew that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)\times(x+z)$; and find the three factors of $x^3-2x^2-23x+60$.

4. Extract the square root of—

$$(a-b)^2\ \{(a-b)^2-2(a^2+b^2)\}\ +2(a^4+b^4).$$

5. Solve the equation $\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}$.

6. A number consists of three digits, the right hand one being zero. If the left hand and the middle digits be interchanged, the number is diminished by 180; if the left hand digit be halved, and the middle and right hand digits be interchanged, the number is diminished by 336; find the number.

1888-89.

1. Find the factors of the following expressions:—

(i) $2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4$.

(ii) $x^4-(p^2+2)x^2y^2+y^4$.

2. Given the relation $\frac{1-2bx+b^2}{1-b^2}=\frac{1-b^2}{1+2by+b^2}$;

Prove that $\frac{x-y}{1-xy}=\frac{2b}{1+b^2}$.

3. Divide $1+a+a^2+a^3+a^4+a^6+a^7+a^8+a^9+a^{15}$ by $1-a^5+a^6$.

4. Simplify the fraction :—

$$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}.$$

5. Solve the equation :—

$$\frac{a-b}{x}+\frac{a+b}{y}=\frac{2(a^2+b^2)}{a^2-b^2}; \quad \frac{a+b}{x}+\frac{a-b}{y}=2.$$

6. A number consists of three digits whose sum is 10. The middle digit is equal to the sum of the other two; and the number will be increased by 99 if its digits be reversed. Find the number.

7. If 19 lbs. of gold weigh 18 lbs. in water, and 10 lbs. of silver weigh 9 lbs. in water; find the quantity of gold and silver in a mass of gold and silver weighing 106 lbs. in air and 99 lbs. in water.

1889-90.

1. Find the divisor when $(4x^2+5y^2)^2$ is the dividend, $8(x+2y)^2$ the quotient, and $y^2(9x+11y)^2$ the remainder.

2. Find the value of the expression $x(y+2)+\frac{x}{y}+\frac{y}{x}$ in terms of a when $x=\frac{y}{y+1}$ and $y=\frac{a-2}{2}$.

3. Simplify :—

$$\frac{\left(p+\frac{1}{q}\right)^m\left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m\left(q-\frac{1}{p}\right)^n}$$

4. If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$.

5. Solve the following equations :—

(i) $$\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}=\frac{2a^4}{x(a^4+a^2x^2+x^4)}$$

(ii) $$\frac{2x+3}{x+1}=\frac{4x+5}{4x+4}+\frac{3x+3}{3x+1}.$$

6. Show that if a number of two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum.

7. A certain resolution was carried in a debating society by a majority which was equal to one-third of the number of votes given on the losing side; but if with the same number of votes, 10 more votes had been given to the losing side, the resolution would only have been carried by a majority of one. Find the number of votes given on each side.

1890-91.

1. Find the value of—

$$\frac{x^3-3abx-2b^3}{x^2-ab}+\frac{x^2-4ab}{x-2a} \text{ when } a=4+b.$$

2. (*a*) If $m=a^x$, $n=a^y$, and $a^2=(m^y n^x)^z$, shew that $xyz=1$.

(b) Simplify :—

$$\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2+(3y-4z)^2}{(4z+2x)^2-9y^2}$$

3 Extract the square root of $\frac{x^2}{a^2}-\frac{2x}{a}+3-\frac{2a}{x}+\frac{a^2}{x^2}$.

4. Find the L.C.M. of :—

x^3-3x^2+3x-1; x^3-x^2-x+1; and x^4-2x^3+2x-1.

5. Solve :—

(i) $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$.

(ii) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6$; $\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=8$, $\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=10$

6 Two vessels contain mixtures of wine and water; in one there is twice as much wine as water, and in the other, three times as much water as wine. Find how much must be drawn off from each, to fill a third vessel, which holds 15 gallons, in order that its contents may be half wine and half water

1891-92.

1 Simplify by using factors :

(i) $\frac{x^2-7xy+12y^2}{x^2+5xy+6y^2}\div\frac{x^2-5xy+4y^2}{x^2+xy-2y^2}$.

(ii) $\frac{(x^2-xy+y^2)^2+(x^2+xy+y^2)^2}{2(x^2+y^2)}$.

2. (a) If $a^2-b^2=b^2-c^2=c^2-a^2$, shew that

$$\frac{ab-c^2}{a-b}+\frac{bc-a^2}{b-c}+\frac{ca-b^2}{c-a}=0.$$

(b) Simplify :—

$$\frac{\left(p^2-\frac{1}{q^2}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q^2-\frac{1}{p^2}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}.$$

3. Find the G.C.M. and the L.C.M. of :—

$x^3-2x^2-19x+20$; $x^3+2x^2-23x-60$; $x^4+7x^3-4x^2-52x+48$.

4. Solve $\cfrac{6}{7-\cfrac{6}{7-\cfrac{6}{7-x}}}=1$.

5. What value of a will make the product of $3-8a$ and $3a+4$ equal to the product of $6a+11$ and $3-4a$?

6. The gross income of a certain man was £40 more in the second of two particular years than in the first, but in consequence of the income tax rising from 4*d.* in the pound to 6*d.* in the pound in the second year, his net income after paying the tax was unaltered. Find his income in each year.

7. The sum of the ages of a man and his wife is six times the sum of the ages of their children. Two years ago the sum of their ages was ten times the sum of the ages of the children and six years hence the sum of their ages will be three times the sum of the ages of the children How many children have they?

1892-93.

1. Define an 'algebraical expression' and the 'degree of an expression.' What is a homogeneous expression?

Find the numerical value of $(xyz+x+y+z)^2-(xy+yz+zx+1)^2$ when $x=2$; $y=3$; and $z=4$.

2. Find the G.C.M. of $2x^3-x^2-x-3$ and $x^5-x^3-4x^2-3x-2$; and the L.C.M. of $6x^2-4$; $4x^2-36$; $3x^2-7x-6$ and $3x^2+7x-6$.

3. Simplify –

(i) $$\sqrt{\left\{\frac{(a-b)^6+8ab(a-b)^4+16a^2b^2(a-b)^2}{a^6b^2-2a^4b^4+a^2b^6}\right\}}.$$

(ii) $$x+1-\cfrac{x}{x+2-\cfrac{x+1}{x+\cfrac{1}{x+2}}}.$$

4. Solve

(i) $$\tfrac{1}{2}(x-2)-\frac{x-5}{9}+\frac{5(x-1)}{6}=0.$$

(ii) $$\frac{3}{x}+\frac{6}{y}=4\,;\ \frac{9}{x}-\frac{2}{y}=2.$$

(iii) $$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$$

5. Find the fraction which becomes equal to a half when the numerator is increased by one, and equal to a third when the denominator is increased by one.

6. In a mile race between a bicycle and a tricycle, their rates were proportional to 5 and 4. The tricycle had a minute's start, but was beaten by 176 yards. Find the rates of each.

APPENDIX III.

MADRAS UNIVERSITY.

MATRICULATION EXAMINATION PAPERS.

1857.

1 Find the value of $a^3+b^3+c^3-3abc$, when $a=1$, $b=4$, $c=-5$.

2. Divide x^2+xy+y^2 by $x+\sqrt{xy}+y$.

3. Solve (1) $\frac{x}{5}+\frac{x-3}{4}=\frac{x+1}{2}-\frac{x-1}{7}$.

(2) $x(y+z)=22$, $y(x+z)=40$, $z(x+y)=42$.

4. A train running from A to B meets with an accident 50 miles from A, after which it moves with $\frac{3}{5}$ths of its original velocity and arrives at B 3 hours late. Had the accident happened 50 miles further on, it would have been only two hours late. Find the distance from A to B and the original velocity of the train.

1858.

1. Find the numerical value of $\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$, when $a=1$, $b=3$, $c=5$.

2. Expand and simplify the quantities in the preceding question.

3 Find the square of $1+2x-x^2-\frac{1}{2}x^3$.

4. Solve $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$; $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$.

5. A number of two digits when divided by their sum gives the quotient 4; but if the digits be inverted, and the number thus formed be increased by 12 and then divided by their sum the quotient is 8. Find the number.

6. Explain what is meant by $a:b::c:d$ and from the principles here involved deduce the common Rule of Three.

1859.

1. Prove that $-a\times-b=+ab$; and that $a^{-\frac{1}{2}}=\frac{1}{\sqrt{a}}$.

2. Divide $1-\frac{x}{2}$ by $-\frac{x}{3}-\frac{x^2}{4}$ to 5 terms.

3. Resolve the following Algebraic quantities into their elementary factors: x^6-a^6; $x^2-9x+14$.

4. Solve (i) $6x-a : 4x-b :: 3x+b : 2x+a$.

(ii) $x-y-z=4$; $3y-x-z=16$; $7z-y-x=22$.

5 A can do a piece of work in 12 days; when he has been at work 4 days, B is sent to help him and they finish it together in 3 days. In how many days could B do the whole?

1860.

1. Simplify the following:—

$$\left\{\frac{(x-a)(y-b)\sqrt{xy.x^2}}{\sqrt[3]{by}\sqrt[3]{a+b}\,(x-y)^7}\right\}^0 \times \frac{(x+y)(x-y)(x^2+y^2)}{x^4-y^4}$$

$$\times\sqrt{a^2+2ab+b^2}.$$

2 Divide $x^4+3x^3y-x^2(y-6y^2)-3xy^2+(15x-1)y^3+5y^4$ by $x^2+3xy+y^2$.

3. Prove that $x^{-6}=\frac{1}{x^6}$.

4. If $a : b :: c : d$, shew that if x be homogeneous with a, b, c, and d, $a^3+x^3 : b^3c^3 :: 1+\frac{x^3}{a^3} : d^3$.

5. In the following equation show that the value of x is independent of a:—

$$\frac{x+a}{a+b}+\frac{x+b}{a-b}=\frac{(a+b)^2}{a^2-b^2}.$$

6. Find x and y from the following equation.—

$$\sqrt{x}+\sqrt{y}=2 \text{ and } x+y=3.$$

7. Two years after the flood, when Shem had lived a sixth of his life, he begat Arphaxed; who, when he was 35 years old, begat Salah. When Salah had lived a twentieth of Shem's life, he begat Eber, who after 34 years begat Peleg. Peleg begat Reu at the same age as Salah begat Eber. Reu was two years older when he begat Serug. Serug begat Nabor at the same age as Peleg begat Reu. At a year younger Nabor begat Terah, who, when he had lived $\frac{7}{60}$ths of the years of Shem's life, begat Abram. Abram was born 292 years after the flood and lived 175 years. By how many years did Abram survive Shem or Shem, Abram?

1861.

1. Multiply $x^4+7x^3y-8x^2y^2-13xy^3+5y^4$ by $6x^2y-3y^2x$.

2. What is the numerical value of $\frac{x-y}{4}\times\frac{z-4}{x}+\frac{x}{y}-\frac{y}{z}$,

if $x=6$, $y=3$, $z=5$?

3. Simplify $\sqrt[6]{\left\{\left(\frac{x^4-a^4}{x^2+a^2}\right)^3\right\}^4}$ and $\left\{\left(\sqrt{x}\cdot\sqrt{y^3}\right)^4\right\}^{\frac{1}{3}}$.

4. Separate into its factors $a^4+2a^3x-2ax^3-x^4$.

5. Find the cube root of $\frac{1}{8}a^3-\frac{1}{4}a^2b+\frac{1}{6}ab^2-\frac{1}{27}b^3$.

6. Add together $\frac{4a+6b}{a+b}+\frac{6a-4b}{a-b}-\frac{4a^2+6b^2}{a^2-b^2}+\frac{4b^2-6a^2}{a^2+b^2}+\frac{20b^4}{a^4-b^4}$.

What will be the value of the sum (1) if $a=b$, (2) $a=-b$, (3) $a=b(2+\sqrt{5})$, (4) $a=2$, $b=0$?

7. At the last Matriculation Examination, a fourth part of the candidates and six more passed in the second division, a thirteenth part in the first, and three more than half failed. How many candidates were examined?

8. A man has in his purse sovereigns and shillings. If he receive as many sovereigns as he has in his purse and pay away his shillings and an equal number of sovereigns, he will have 6 coins. But if he double the number of his shillings, retaining the original number of sovereigns, he will have 9 coins. How many sovereigns and how many shillings were in his purse at first?

9. Solve the following equations:—

(1) $\frac{x-4}{x-2}+\frac{x-3}{x-5}=2$. (2) $\frac{x-a}{a}+\frac{x-b}{b}+\frac{x-c}{c}=1$.

(3) $\frac{x}{ab}+\frac{x}{bc}+\frac{x}{ca}=a+b+c$. (4) $\sqrt{x^2+4x+4}=3x-5$.

1862.

1. Add together $2a^3+4a^2+6ab$, $a^2+b^2+ab^2+b^3$; $ab^2-4ab-2a^2$, $3a^2b-a^3$, ab^2+a, $b-2a^2$, and reduce the sum to its simplest form.

2. Divide $a^2b^2x^2y^2-a^2cx^2y+3b^3xy^2+3bcxy+(b^2cy-c^2)y$ by b^2y^2-cy, and square the quotient.

3. What are the numerical values of $\sqrt[3]{2a^3+a^2+a}$, $\sqrt[3]{2a^3+a^2-a}$, and $\sqrt{\frac{1}{a^3}+\frac{1}{a^2}-\frac{2}{a}}$, when $a=\frac{1}{2}$?

4. What are the naught, first, second, and fourth powers of $\sqrt{4a^2-4ax^2+x^4}$? What is the square of the quantity whose square root is a?

5. Find the square root of $16m^2x^2+36n^2y^2+9p^2z^2+48mnxy+36npyz+24mpxz$.

6. Solve the following equations. (1) $x^2-x-12=(3+x)(31-4x)$;

(2) $\frac{ax}{b}+\frac{bx}{c}+\frac{cx}{a}=\frac{(ab^2+bc^2+ca^2)^2}{a^2b^2c^2}$; (3) $2x+3y=5x-\frac{1}{2}=4y+\frac{5}{3}$.

7. Two trains start at the same time from A and B for the C junction. The train from A should run at 24 miles an hour, and reach the junction half an hour before that from B, which travels 18 miles an hour. But the former is so retarded as only to run at an average rate of 22 miles an hour. The two trains arrive at the junction at the same time. How far are A and B respectively from C; and how long were the trains upon the road ?

8. A and B went out to shoot. A shot three pheasants for every 5 partridges; B 5 pheasants for every 9 partridges; A shot four birds to B's 5. How many pheasants, and how many partridges had they brought down when they had shot 126 birds ?

1863.

1. Simplify $(a+b+c)-(a-b+c)\ \{(a+c)^2-b^2-(a^2+b^2+c^2)\}$.

2. Multiply (1) x^2+3x+9 by x^2-3x+9. (2) $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

3. Divide (1) x^3-7x-6 by $x-3$. (2) $a^{-3}+b^{-3}$ by $a^{-1}+b^{-1}$.

4. Define the square root of any quantity.

5. Find the square root of (1) $x^6+14x^5-4x^4-28x+4x^2+49$.
(2) $a+b+c+2\sqrt{ab}-2\sqrt{bc}-2\sqrt{ac}$.

6. Find the cube of $a+b-c$, using factors and not direct multiplication.

7. Solve (1) $\dfrac{7x+5}{3}-\dfrac{24-4x}{5}+5=\dfrac{3x+7}{2}$.

(2) $\dfrac{x-a}{b}-\dfrac{x-b}{c}-\dfrac{x-c}{a}=\dfrac{x-(a+b+c)}{abc}$.

(3) $\left.\begin{array}{l} ax+by=c \\ a_1x+b_1y=c_1 \end{array}\right\}$ (4) $\left.\begin{array}{l} 3x+2y=1{\cdot}4 \\ 5x-{\cdot}1y=1{\cdot}99 \end{array}\right\}$.

(5) $\dfrac{1}{x}+\dfrac{1}{y}=3$; $\dfrac{1}{y}+\dfrac{1}{z}=4$; $\dfrac{1}{z}+\dfrac{1}{x}=5$.

8. A merchant goes to three bazaars in succession. At the first he gains 15 per cent. on his capital; at the second 20 per cent. upon this increased capital; and at the third 25 per cent. on what he then possessed; on his return home he finds that he has gained Rs. 2,639. What was his original capital ?

9. A certain fraction becomes $\frac{1}{2}$ when 1 is added to its denominator and [illegible] when 2 is taken away from its numerator. What is the fraction ?

1864.

1. If $a=1$, $b=2$, $c=3$, $d=0$; find the value of

$$\frac{a^2b+b^2c+c^2d+d^2a}{(a+b)(c+d)-\{(a-d)+(c-b)\}}.$$

(2) $\sqrt[3]{b-a}+\sqrt[3]{4(c-a)}-\sqrt[3]{3(8a+5b+3c-2d)}$.

2. Simplify $\{2x^2-y(3x-y)+x+(y-1)\}$
$-\{2(x+y)+3(y+1)(y-1)+2x(x-2y)\}$
$+\Big((x-y+1)-\big(xy-2(y^2+y-2)\big)+2\Big)$

3 Multiply $a^{-3}-4a^{-2}b-4a^{-1}b^2-b^3$ by $a^{-2}-2a^{-1}b+b^2$

4. Divide $(a^2-b^2)x^2+(a^2-b^2)y^2-2(a^2+b^2)xy$ by $(a+b)x-(a-b)y$ without removing the brackets.

5 Find (1) the square root of $a^2+4b+9\sqrt[3]{c^2}+4\sqrt{a^2b}-6a\sqrt[3]{c}-12\sqrt[6]{b^3c^2}$.

(2) the cube root of 28934443.

6. Find the G.C.M of $a^5-3a^4b+4a^3b^2-5a^2b^3+3ab^4-2b^5$ and $2a^4-3a^3b-a^2b^2-2ab^3$.

7. Reduce the following to their simplest forms

(1) $\dfrac{x^2+x-12}{x^2+7x+12}$ (2) $\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}$.

(3) $\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right)\div\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)$.

8 Solve (1) $\dfrac{x^2-a}{x-\sqrt{a}}+\dfrac{x-a}{x+\sqrt{a}}=x-\dfrac{x-\sqrt{a}}{2}$

(2) $2x-a-\dfrac{y-b}{a+b}=a$; $2y-b+\dfrac{x-a}{b}=2b-y$

(3) $xyz=(xy+xz-yz)=4(yz+xy-xz)=6(xz+yz-xy)$

9. A merchant speculated in 20 voyages On examining his accounts he found that, on an average, in each prosperous voyage, he had gained a sum equal to $\frac{1}{6}$ of his original capital, and that in each adverse voyage he had lost a sum equal to $\frac{1}{4}$ of the same On the whole, however, his capital was increased by $\frac{5}{12}$ of itself. How many prosperous voyages did he make?

February 1865.

1. Prove that $(a^2+ab+b^2)(c^2+cd+d^2)=(ac+ad+bd)^2+(ac+ad+bd)(bc-ad)+(bc-ad)^2$

2. Divide (1) $2x^3-9ax^2+11a^2x-6a^3$ by $x-3a$;

(2) $8x-8x^{\frac{2}{3}}y^{\frac{1}{3}}+4x^{\frac{1}{3}}y^{\frac{2}{3}}-y$ by $2x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

3. Find the square root of (1) $a^2+\dfrac{1}{a^2}-2\left(a+\dfrac{1}{a}\right)+3$,

(2) $a^{-4}+b^{-4}-a^{-2}b^{-2}+2a^{-3}b^{-1}-2a^{-1}b^{-3}$.

4. Find the G.C.M. of $4x^4+9x^3+2x^2-2x-4$ and $3x^3+5x^2-x+2$.

5. Find the L.C.M. of $3x^2-5x+2$ and $4x^3-4x^2-x+1$.

6. Shew (1), that if P divide A, it will also divide mA; and (2), that if P divide A and B, it will also divide $mA \pm nB$.

7. Simplify (1) $\dfrac{3a^4-a^2b^2-2b^4}{10a^4+15a^3b-10a^2b^2-15ab^3}$;

(2) $\dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}}$.

8. Solve (1) $\dfrac{6x+18}{13}-4\tfrac{5}{8}-\dfrac{11-3x}{36}=5x-48-\dfrac{13-x}{12}-\dfrac{21-2x}{18}$;

(2) $\dfrac{3-3x}{2}-\dfrac{2-2x}{2}=\dfrac{1-x}{4}$; (3) $\dfrac{x-1}{\sqrt{x}+1}=1+\dfrac{\sqrt{x}-1}{2}$;

(4) $5x+2y+z=30$; $\dfrac{x}{2}+\dfrac{4y}{5}-\dfrac{z}{10}=4$; $2x+5y+10z=129$.

9. A and B start together on a certain journey. When they have walked a distance of a miles, A finds it necessary to return home, and goes at twice his former rate. He then starts again at $\dfrac{m}{n}$ times his original pace and just at the end of the journey overtakes B, who since A left him had gone at $\dfrac{n}{m}$ times the original pace. How long was the journey?

December 1865.

1. Find the continued product of

$x+\sqrt{3x}+3$, x^2-3x+9 and $x-\sqrt{3x}+3$.

2. Simplify (1) $\dfrac{a^2}{(x-a)^n}+\dfrac{2a}{(x-a)^{n-1}}+\dfrac{1}{(x-a)^{n-2}}$.

(2) $\dfrac{a+b-c}{c-a}-\dfrac{a+b+c}{c-b}+\dfrac{(a+b-c)(a-b)}{c^2-(a+b)c+ab}$.

3. Resolve each of the following expressions into three factors.

(1) $b^3-a^3-(c^2-ab)(b-a)$; (2) $a(b^2+c^2-a^2)+b(a^2+c^2-b^2)$.

4. Find the value of $\dfrac{(x-a)(x-b)}{(x-a-b)^2}$, when $x=\dfrac{a^2+ab+b^2}{a+b}$.

5. Shew that $\left(\dfrac{1}{a}-\dfrac{1}{c}\right)^2+\dfrac{4}{(a+c)^2}=\left(\dfrac{a+c}{ac}-\dfrac{2}{a+c}\right)^2$.

6. Find the L.C.M. of $x^3-7x-36$, $x^3-13x-12$ and x^2+4x+3.

7. Solve (1) $\dfrac{3x+5}{x+1}=\dfrac{4x+8}{3x+3}+\dfrac{10x+1}{6x+3}$; (2) $\sqrt{x^2+9x}-\sqrt{x^2-8x}=\sqrt{x}$.

(3) $ax+by+cz=a+b+c$; $\frac{ax}{b+c}+\frac{by}{a+c}=1$; $\frac{2x}{b+c}+\frac{ay}{a+c}=\frac{1}{a}+\frac{1}{b}$.

8. A, B and C work together at building a wall for 10 days, after which B stops working, and A and C together finish it in 5 days. Find the time in which each can build it separately, if A and B together can do as much in a day as C can do in 3 days, and 3 days' work of B is equal to 4 days' work of C.

1866.

1. Reduce to its simplest form $\frac{x^3-1}{x^3+1}+\frac{x^3+1}{x^3-1}-\frac{2x^4}{x^4+x^2+1}$.

2. Divide

$$(a+b)(b+c)x^5+\{(b+c)^2+(a+b)^2\}x^4+(b+c)(a-b)x^2+(a+b)\times(a+c)x-(a+c)(b+c) \text{ by } (b+c)x^2+(a+b)x-(b+c).$$

3. Find the square root of

$$a^4+\frac{1}{a^4}+a^2+\frac{1}{a^2}+2\left\{a^3-\frac{1}{a^3}-\left(a-\frac{1}{a}\right)\right\}.$$

4. Find the G.C.M. of x^5-2x^4-x-1 and x^5+2x^3+x+1.

5. Prove that every common multiple of a and b is a multiple of their L.C.M.

6. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{a^3}{b^3}=\frac{pa^3+qc^3+re^3}{pb^3+qd^3+rf^3}$.

7. Solve (1) $\frac{x-1}{3}+\frac{1\frac{1}{2}-2x}{4\frac{1}{2}-5x}=\frac{\frac{1}{2}(2x-1)}{2}+2-\frac{7}{18x+6}-\frac{7x}{6x+2}$;

(2) $x+y=2(z+1)$, $y+z=x+1$, $z+x=3y+1$.

8. If $\frac{a}{b}+\frac{c}{d}=\frac{b}{a}+\frac{d}{c}$, prove that $\frac{a^3}{b^3}+\frac{c^3}{d^3}=\frac{b^3}{a^3}+\frac{d^3}{c^3}$.

9. A, B, C start at the same instant from P to run to Q, their rates being such that B is always as much behind A as he is in advance of C. After A has reached Q, he returns at once to P at the same rate and meets B at a point whose distance from Q is equal to one-fourth of PQ. Shew that A meets C at a distance from P equal to one-third of PQ.

1867.

1. Divide $(m+n)a^4+(m^2-1+n^2+2mn)a^3-(m+n)(1-2mn)a^2+(m-2mn+n)a-1$ by $(m+n)a-1$ without removing the brackets.

2. If $x+\frac{1}{x}=2(a+m)$, $x-\frac{1}{x}=2b$, $y+\frac{1}{y}=2(c+n)$, $y-\frac{1}{y}=2d$,

find the value of $xy+\frac{1}{xy}$.

3. Simplify (a) $\frac{ab}{a+b}\left(3c+\frac{b}{a}\right)-\frac{b}{(a+b)^3}(a^2+b^2)-2a\left(\frac{b}{a+b}\right)$

(b) $$\frac{\frac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\frac{1}{b-a}-\frac{1}{a+b}} \div \frac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2}.$$

4. Find the G.C.M. of x^4-4x^2+5x-4 and x^4-x^3+x-1.

5. If $ap=bq=cr$, shew that $\frac{p^2}{qr}+\frac{q^2}{pr}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2}$.

6 Find the L.C.M. of x^4+4x^2-5, x^2-4x+3, and $2x^3+x^2-8x+5$.

7. Solve (a) $\frac{6-5x}{5}-$ [illegible] $\frac{7-2x^2}{x-1}-1\frac{1}{10}=\frac{1+3x}{7}$ [illegible] $+\frac{1}{15}$;

(b) $\frac{7abc}{a+b+c}+\frac{(a^2+b^2+c^2)c}{(a+b+c)^2}+\frac{2abc(ab+ac+bc)}{(a+b+c)^3}=8c$.

(c) $(a-b)x+(b-c)y+cz=1$; $2ax+by+\frac{2}{3}cz=2$; $(a+b-c)x+(a-2b+2c)y+2bz=3$.

8. A alone can do a piece of work in a hours; A and C together can do it in b hours; and C's work is $\frac{1}{n}$th of B's. The work has to be completed in c hours. Find (i) how long after A has commenced, B and C should *relieve* him, so as to finish the work in time; (ii) how long after A has commenced, B and C should *join* him, so that the three working together might just complete the work in time.

1868.

1. Divide $bc(c-b)+ac(a-c)+ab(b-a)$ by $(a-b)(a-c)$.

2. Resolve the first of the following expressions into 2 factors and the second into 4 factors: (1) x^4+x^2+1; (2) $a(b^3-c^3)+bc(c^2-b^2)+a^3(c-b)$.

3. State in what cases x^n+a^n will be divisible by $x+a$ and x^2+a^2 respectively; state also the number of terms in the quotient in each case. Show that the last digit in $3^{2n+1}+2^{2n+1}$ is 5 whatever n may be.

4. Simplify the following expressions:

(1) $\frac{1}{x^2+x+1}+\frac{2x}{x^4+x^2+1}$;

(2) $\frac{b+c-2a}{b-a}-\frac{(b-c)^2}{(a-b)(a-c)}+\frac{b+c-2a}{c-a}$.

5. Find the G.C.M. of x^3-7x+6 and $6x^3-7x^2+1$.

6. If $x=\frac{2ac}{a+c}$, shew that the value of $\frac{(x-a)^2+(x-c)^2}{a^2+c^2}+\frac{4ac}{(a+c)^2}$ is the same for all values of a and c.

7. Solve (1) $\frac{1-4x}{5}+\frac{5}{9}-\frac{3-7x}{4}=x$;

(2) $ax-by=a-b$; $\frac{x}{2a}+\frac{y}{2b}=\frac{1}{a+b}$;

(3) $\sqrt{y}-\sqrt{y-2x}=\sqrt{48-2x}$; $y(x-15)=36$.

8 A room of which the floor is rectangular is such that the addition of a foot to the height will increase the area of the walls as much as the addition of a foot to both the length and breadth, the increase in each case being 60 square feet; and if the floor be made square, the perimeter remaining the same as before, its area will be increased by 9 square feet. Find the length, breadth, and height of the room.

1869.

1. Simplify $(\sqrt{1-a^2}\cdot\sqrt{1-x^2}+ax)^2-2ax(\sqrt{1-a^2}\cdot\sqrt{1-x^2}+ax)+x^2$.

2. Divide $x^{10}-a^5x^5+a^{10}$ by x^2-ax+a^2.

3. Shew that b^2-ac is a factor of $(2b^2+a^2-ac)(2b^2+c^2-ac)-b^2(a+c)^2$ and the other factor is positive for all values of a, b and c.

4. Simplify (1) $\dfrac{1+\frac{1}{2}\cdot\frac{x}{a}}{\left(1+\frac{x}{a}\right)^2}+\dfrac{1+\frac{3}{2}\cdot\frac{a}{x}}{\left(1+\frac{a}{x}\right)^2}$.

(2) $\frac{1}{(x+2)(x+4)}+\frac{1}{(x+1)(x+3)}-\frac{3}{(x+1)(x+2)(x+3)(x+4)}$.

5. Arrange the expression $(b+c)^2(a+d)^2-4(ab+cd)(ac+bd)$ according to powers of b and hence find its square root.

6 Shew that the two expressions $(a^2-a+1)(b-c)+(b^2-b+1)(c-a)+(c^2-c+1)(a-b)$ and $(a^2-a+1)(b^2-c^2)+(b^2-b+1)(c^2-a^2)+(c^2-c+1)(a^2-b^2)$ are equal.

7. Prove the rule for finding the G.C.M. of two Algebraical expressions. Shew that $\frac{2x^3+5x^2+19}{x^3+5x+12}$ is in its lowest terms

8. If $\frac{a-1}{x}-\frac{a-2}{y}=\frac{1}{b}$ and $\frac{b-1}{x}-\frac{b-2}{y}=\frac{1}{a}$ shew that $\frac{c-1}{x}-\frac{c-2}{y}=\frac{c}{ab}$.

9. Solve (1) $\frac{12x+19}{18}-\frac{7x-2}{3x-10}=\frac{8x-25}{12}+\frac{5}{36}$;

(2) $a\sqrt{\dfrac{1+x}{1-x}}+(a-2)\sqrt{\dfrac{1-x}{1+x}}=2\sqrt{a(a-2)}$;

(3) $x+y+z=(a-b)(b-c)(a-c)$; $ax+by+cz=0$; $a^2x+b^2y+c^2z=0$.

10. A horseman travelling at a walking pace of 4 miles an hour meets a bandy going in the opposite direction at the rate of 2 miles an hour; after proceeding at the same pace for half an hour he turns and canters back till he overtakes the bandy. If he had continued for another quarter of an hour before turning, the bandy would have been $\frac{3}{8}$ ths of a mile further on, before it was overtaken. Find the rate at which the horseman cantered.

1870.

1. Divide the difference between $(x+a)(x+b)(x+c)$ and $(y+a)(y+b)\times(y+c)$ by $x-y$.

2. Find in the simplest form the difference of the squares of

$$ab+\sqrt{(a^2-1)}\sqrt{(b^2-1)} \text{ and } b\sqrt{(a^2-1)}+a\sqrt{(b^2-1)}.$$

3. Simplify (1) $\dfrac{ac-1}{(a-b)(1+ax)}+\dfrac{bc-1}{(b-a)(1+bx)}$; (2) $\dfrac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$.

4. Find the greatest common measure of x^4-x^3+7x+5 and x^4-x^3+2x+2.

5. If $x=\dfrac{(2c-b)^2-ab}{a+3b-4c}$, find the value of $\sqrt{\left(\dfrac{x+a}{x+b}\right)}$.

6. Prove that every common multiple of two numbers is a multiple of their least common multiple. Find the least common multiple of x^5-1 and x^7-1.

7. Shew that

$$\left\{\frac{(mn+1)^2}{m^2n^2}-\frac{4}{mn}\right\}y^2+2\left\{\frac{(mn+1)^2}{mn}-2mn-\frac{2}{mn}\right\}y+(mn+1)^2-4mn$$

is a perfect square, and find its square root.

8. If $x^2+y^2=1$, $2xy=\dfrac{a-b}{a+b}$, and $2nx=x+y$, shew that $\left(1-\dfrac{1}{n}\right)^2a=b$.

9. Solve the equations

(1) $\dfrac{1}{x-3}-\dfrac{1}{x-4}=\dfrac{3}{3x-13}-\dfrac{3}{3x-16}$;

(2) $\sqrt{(a+x)}-\sqrt{(a-x)}=\sqrt{nx}$.

(3) $ax-by=\dfrac{b-a}{2}$; $ax+by=c(1+z)$; $by-cz=\dfrac{c-b}{2}$.

10. A and B start from opposite ends of a straight course each walking uniformly; (A, who is the faster walker, at the rate of four miles an

hour), and meet at the end of two hours. If, when A reached the middle point of the course, they had interchanged their rates of walking, they would have met a quarter of a mile nearer the middle point. Find B's rate of walking, and the length of the course.

1871.

1. If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{a^2+c^2-b^2}{2ac}$, and $x=\dfrac{a^2+b^2-c^2}{2ab}$, find in the simplest form the values of (1) $(b+c)x+(c+a)y+(a+b)z$; (2) $\dfrac{x+yz}{y+xz}$.

2. Reduce to their simplest forms

(i) $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-x^4}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{x^4-(x+1)^2}$;

(ii) $\dfrac{x-1}{x^2+x+1}+\dfrac{x+1}{x^2-x+1}-\dfrac{2x(x^2-2)}{x^6-1}$;

(iii) $$\frac{\dfrac{1+\dfrac{a-bx}{c+bx}}{x-2(a+c)}}{1+\dfrac{x}{a+c-x}}-\frac{\dfrac{1+\dfrac{a-bx}{c+bx}}{x-2(a+c)}}{1+\dfrac{x}{a+c-x}}.$$

3. If a quantity measure two other quantities, prove that it will also measure the sum or difference of any multiples of those two quantities.

If the two expressions $ax^3-c(3a+b)x^2+(a^3+bc^2)x+d$ and $bx^3+c(a-b)x^2+a(c^2-a^2)x-d$ have a common quadratic factor, (that is a factor containing x^2 as the highest power of x), prove that this factor is an exact square.

4. Find the square root of $x^2-2+\dfrac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}$.

5. There are two quantities a and b of which the L.C.M. is x, and the G.C.M. is y; if $x+y=ma+\dfrac{b}{m}$, prove that $x^3+y^3=m^3a^3+\dfrac{b^3}{m^3}$.

6. Find the G.C.M. of $x^8+36x^3-4x^2+8x-17$ and $x^6+8x^3-x^2+2x-4$.

7. If $ab+ac+bc=1$, prove that

$$\left\{1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right\}^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}.$$

8. Solve (i) $\dfrac{10x+47}{18}-\dfrac{12x+38}{13x+23}=\dfrac{5x+11}{9}$;

$$\left.\begin{aligned}\text{(ii) } (a+b)x-(a-b)y&=\frac{3a+7b}{21}(x^2-y^2)\\ (a-b)x+(a+b)y&=\frac{7a-3b}{21}(x^2-y^2)\end{aligned}\right\};$$

$$\text{(iii) } \frac{x+2y+1}{2}=\frac{y+3z+2}{3}=\frac{z+4x+3}{4}=2.$$

9. A and B play four games of chance of which A wins the first and last, and B the other two. The amount which each stakes for the first game is half the whole sum of money possessed by both together, and for the other games half the money possessed by the loser of the preceding game. At the end of the 4th game, A finds that he has 18 shillings less than he would have had if he had won them all, and B finds that he has 9 shillings less than he had at starting. Find the amount of money possessed by each at first.

1872.

1. Divide the difference of $(x^2-bx+b^2)(x+a-b)$ and $(x^2-ax+a^2)\times(x-a+b)$ by $a-b$.

2. For what values of n is x^n+a^n divisible by $x+a$, and x^n-a^n by $x-a$? Write down the last three terms of the quotient in each case.

Shew that (i) $4.2^{4n}+2^{2(n+1)}+1$ is divisible by 9;

and (ii) $(1-x)^{2n}-(4-7x-x^2)^n$ both by $2x-1$ and $x+3$, n being any positive integer.

3. Simplify (i) $\dfrac{3}{2(1+x)}-\dfrac{5}{2(1-x)}+\dfrac{6}{1-2x}$;

(ii) $\dfrac{bc-a^2}{(b^2-ac)(ax-b)}+\dfrac{c^2-ab}{(ac-b^2)(bx-c)}$.

Shew that the numerator of (1) in its simplest form is the sum of two squares.

4. In the process for finding the G.C.M. of two quantities A and B, Q is any remainder and P the preceding divisor: shew that the G.C.M. of Q and $P-nQ$ will be the G.C.M. of A and B, where n is any factor arbitrarily chosen, and not necessarily the quotient arising from the division of P and Q. In what cases will this principle facilitate the process?

5. Find the G.C.M. of $6x^3-7x^2+1$ and $3x^4-8x^3-12x^2+1$.

6. Shew that $\dfrac{(a^2+b^2)(1+a^2b^2)}{a^2b^2}$ is the sum of two squares.

7. The remainder after finding the first two terms of a square root of the form ax^2+bx+c is $-6x^2+4x+1$: determine the root.

8. Solve (i) $\dfrac{x-3}{x-5}-\dfrac{x-7}{x-9}=\dfrac{x-11}{x-13}-\dfrac{x-15}{x-17}$;

(ii) $\sqrt{ax-b}-\sqrt{bx-a}=(\sqrt{a}-\sqrt{b})\sqrt{x-1}$;

(iii) $ax-by=\dfrac{3(b-a)}{2}$; $by+2cz=\dfrac{3(a-c)}{2}$; $ax-by+cz=0$.

9. A and B travel in opposite directions from two places C and D, starting at the same time, and meet at a point 10 miles nearer D than C. If each had travelled a mile an hour faster, they would have met an hour sooner; and if A had travelled half a mile an hour slower, and B half a mile an hour faster, they would have met $2\frac{1}{2}$ miles further from D. Find the rate of travelling of each, and the distance between C and D.

1873.

1. Divide $a(a+b)(a+c)-b(b+c)(b+a)$ by $(a-b)$ and by $(a+b+c)$.

2. Assuming that x^n-y^n is divisible by $x-y$, when n is any whole number, shew that $(ab)^n-(bc)^n+(cd)^n-(da)^n$ is always divisible by $ab-bc+cd-da$.

3. Prove that any common multiple of a and b is a multiple of their Least Common Multiple.

4. Find the G.C.M of a^5+2a^2+4 and $a^5-3a^4+2a^3-4$.

5. Simplify (i) $\dfrac{(n+2)^2(n-1)^2-(n-1)^2(n-2)^2}{(n+1)^3+n^3+(n-1)^3}$;

(ii) $\dfrac{1}{(x+1)(2x+1)}+\dfrac{1}{(2x+1)(3x+1)}+\dfrac{1}{(3x+1)(4x+1)}+\dfrac{1}{(4x+1)(5x+1)}$.

6. Shew that $(a-b)^2(c+d)^2+4ab(c^2+d^2)-4cd(a^2+b^2)$ is an exact square.

7. Prove that $\dfrac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}=\frac{3}{2}(a+b+c+d)$.

8. Shew that $\dfrac{ab}{(x-a)(x-b)}+\dfrac{bc}{(x-b)(x-c)}+\dfrac{ca}{(x-c)(x-a)}=0$

when $\dfrac{1}{x}=\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

9. Solve (i) $\dfrac{3}{x-3}-\dfrac{4}{x+9}-\dfrac{5}{x-27}+\dfrac{6}{x-15}=0$;

(ii) $\sqrt{a-m^2x}-\sqrt{b-n^2x}=\sqrt{a+b-(m+n)^2x}$;

(iii) $\dfrac{a}{x}+\dfrac{b}{y}=\dfrac{a}{y}+\dfrac{b}{z}=\dfrac{a}{z}+\dfrac{b}{x}=c$.

10. A gentleman went out for a walk, and, after having been out 12 minutes, was overtaken by his servant who had run from the house at

twice his master's pace. The master then bade the servant run back at the same rate to the house and bring his cigars, while he walked on at his former pace. If the master was one mile from the house when overtaken the second time, at what rate did he walk?

1874.

1. Divide $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3$ by $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

2. Write out the first five terms of the quotient of $\frac{a^{2n+1}+b^{2n+1}}{a+b}$ and $\frac{a^{2n+1}-b^{2n+1}}{a-b}$; and hence find, without multiplication the product of $(a^4+a^3+a^2+a+1)\times(a^4-a^3+a^2-a+1)$.

3. From $(x+a-b)(x+b-c)(x+c-a)$ subtract $(x-a+b)(x-b+c)\times(x-c+a)$ and divide the remainder by $(a-b)(b-c)(c-a)$.

4. Prove that every common measure of two algebraical expressions will divide their G.C.M. Find the G.C.M. of

$$a^5+29a-15 \text{ and } 2a^5-3a^4+16a^2+a-10.$$

5. Simplify (1) $\frac{a^2}{x(a-x)}+\frac{x^2}{a(x-a)}-\frac{(a-x)^2}{ax}$;

(2) $\frac{(ac+bd)^2-(ad+bc)^2}{(a-b)(c-d)}-\frac{(ac+bd)^2+(ad+bc)^2}{(a+b)(c+d)}$.

6. Given that $s=a+b+c$, prove that $(s-3a)^2+(s-3b)^2+(s-3c)^2=3\ \{(a-b)^2+(b-c)^2+(c-a)^2\}$.

7. Extract the square root of

$$(a^2+b^2)(a^2b^2+1)-2ab(a^2-1)(b^2-1)-4a^2b^2$$

8. If $x=\frac{a+1}{a-1}$, $y=\frac{b+1}{b-1}$, $z=\frac{c+1}{c-1}$, shew that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(xy+1)(yz+1)(zx+1)}=\frac{(a^2+1)(b^2+1)(c^2+1)}{(ab+1)(bc+1)(ca+1)}.$$

9. Solve the following equations—

(1) $\frac{x-4}{(x-1)(x-3)}+\frac{x-7}{(x-1)(x-6)}+\frac{x-9}{(x-3)(x-6)}=\frac{3}{x}$;

(2) $\sqrt{(15-9x)}+\sqrt{(10-4x)}=\sqrt{(5-x)}$;

(3) $4ax+(a+1)y=(3a-1)z$; $(3a-1)(x-y-1)+z=0$; $(x+y)=z$.

10. A man rowing against a steam meets a log of wood which is being carried down by the current. He continues rowing in the same direction for a quarter of an hour longer, and then turns and rows down the steam overtaking the log $1\frac{1}{4}$ miles lower down than the point where he first met it. Find the rate at which the current flows.

1875.

1. Remove from brackets $(c-a-b)-\{(b+c-a)-[(a+b+c)-(c+a-b)]\}$.

2. If $x=\frac{a-b}{m-c}$, $y=\frac{b-c}{m-b}$, $z=\frac{c-a}{m-b}$, find the value of $x+y+z+xyz$.

3. Add together the squares of $2\{\sqrt{ab(1+a)(1+b)}+\sqrt{ab(1-a)(1-b)}\}$ and of $\{(a+\sqrt{1-a^2})(b-\sqrt{1-b^2})-(a-\sqrt{1-a^2})(b+\sqrt{1-b^2})\}$ and simplify the result.

4. What must be the form of m in order that a^m-x^m may have both a^n+x^n and a^n-x^n for divisors, n being any positive integer?

Shew that $2^{2n}-1$ is divisible by 15.

Reduce to its lowest terms $\frac{20a^3+11a^2+24}{24a^3+11a^2+20}$.

6. State and prove the rule for finding the Least Common Multiple of the two Algebraical expressions P and Q.

Find the L.C.M. of $1+a+a^3$ and $1+a^4+a$.

7. Simplify (i) $\frac{\sqrt{ax}}{\sqrt{a}+\sqrt{x}-\sqrt{a+x}}-\frac{\sqrt{ax}}{\sqrt{a}+\sqrt{x}+\sqrt{a+x}}$;

(ii) $$\frac{\frac{a+b+c}{a+b-c}+\frac{c+a-b}{b+c-a}}{\frac{a+b-c}{c+a-b}+\frac{b+c-a}{a+b+c}}\div\frac{\frac{a+b+c}{a+b-c}}{\frac{b+c-a}{c+a-b}}.$$

8. Find what term is wanting to make the following expression a complete square: $(a^2x^4+64b^2)-4(ax^2+8b)(a-b)x$.

9. Solve (i) $\frac{x-7}{a-3}+\frac{x-2}{x-9}+\frac{x-4}{x-1}=3$; (ii) $\sqrt{\frac{x-a}{x-b}}+\frac{a}{x}=\sqrt{\frac{x-b}{x-a}}+\frac{b}{x}$;

(iii) $(a^2+b^2)(x-1)=ab(2x-y)$; $4x=y+2$.

10 A person sets out to walk to a certain town. But when he has accomplished a quarter of his journey, he finds that if he continues at the same pace, he will have gone only $\frac{5}{6}$ths of the whole distance when he ought to be at his destination: He therefore increases his speed by a mile an hour, and arrives just in time. Find his rates of walking.

1876.

1. Divide $(a-b)(a+b-c)+(b-c)(b+c-a)$ by $(a-c)$.

2. Separate into four factors:

$$(a^2-b^2)^2+(c^2-d^2)^2-(a+b)^2(c-d)^2-(a-b)^2(c+d)^2.$$

3. In what cases is a^n-b^n divisible by $a+b$ and $a-b$ respectively?

(i) Shew that $(a-1)a^n+(b-1)b^n$ is not divisible by $a+b$;

(iii) and that $(2a+b)^n-(a+2b)^n-a^n+b^n$ is divisible by both $a+b$, and $a-b$ whether n be odd or even.

4. (i) If an Algebraical Expression, is a common measure of two other Algebraical Expressions, prove that it will measure the Sum or the Difference of any multiples of those Expressions.

(ii) If $x^2-(2q-1)x+2q$ and $x^2-2qx-2(q+1)$ have a common factor, determine q; and hence find the L.C.M. of the two expressions.

5. Simplify (i) $(a+b+c)(l+m+n)+(b+c-a)(m+n-l)+(a-b+c)(l-m+n)+(a+b-c)(l+m-n)$;

(ii) $$\frac{a-b}{(x+a)(x+b)}-\frac{b-c}{(x+b)(x+c)}+\frac{c-d}{(x+c)(x+d)}+\frac{d-a}{(x+d)(x+a)}.$$

6. Express the following as the difference of two squares

$$\left\{(a+b)x-(c+d)y\right\}.\left((a-b)x-(c-d)y\right).$$

7. If $(m+a)(b+c)+a^2=(m+b)(c+a)+b^2=(m+c)(a+b)+c^2=n$ determine the values of m and n in terms of a, b, c.

8. Find the cube root of $(x+y)^6-(x-y)^6-12xy(x^2-y^2)^2$.

9. Solve the following Equations:

(i) $$\frac{a-b}{x+c}+\frac{c-d}{x+b}=\frac{a-b}{x+d}+\frac{c-d}{x+a};$$

(ii) $\sqrt{(ax+c)}+\sqrt{(ax+d)}=\sqrt{(bx+c+e)}+\sqrt{(bx+d+e)}$;

(iii) $$\begin{cases} a(x+y)+b(x-y)=a^2-ab+b^2 \\ a(x+y)+b(x-y)=a^2+ab+b^2. \end{cases}$$

10. A letter-carrier has a hours allowed to him for going from A to B and back again, including c hours for rest at B. But he finds that he can get b hours for rest by going d miles an hour faster each way. Find his ordinary speed and the distance from A to B.

1877.

1. Resolve into factors $(2a+2b-ab)^2-(b^2-4a)(a^2-4b)$.

2. Simplify $\frac{a+b}{2ab}(a^2+b^2-c^2)+\frac{b+c}{2bc}(b^2+c^2-a^2)+\frac{c+a}{2ca}(c^2+a^2-b^2)$

and shew that, if $a+b+c=0$: $\frac{a^2+b^2+c^2}{a^3+b^3+c^3}+\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0.$

3. Find the G.C.M. of $x^4-39x-22$ and $11x^4-39x^3-8$ and the square root of $(x^3+4)(x-1)+\left(\frac{x}{2}+\frac{2}{x}\right)^2$

4. Solve $\frac{(x+1)^5+(x-1)^5}{(x+1)^3+(x-1)^3}=10.$

5. A person being asked his age replied—"Ten years ago I was five times as old as my son, but twenty years hence, I shall be only twice as old as he." What is his age?

1878.

1. Simplify $\frac{x^3+3x^2+5x+15}{x^3+2x^2+5x+10}+\frac{x^4+x^3+3x^2+x-2}{x^4+2x^3+3x^2+4x-4}$.

2. Shew that $(x^3-3x)^2-8(x^6-6x^4+9x^2-2)$ is an exact square and resolve the whole expression into factors.

3. Shew that (i) If $a+b+c=0$, then $a(b-c)^3+b(c-a)^3+c(a-b)^3=0$.

(ii) If $a+b+c=1$, $ab+bc+ca=\frac{1}{3}$, $abc=\frac{1}{27}$,

then $\frac{1}{a+bc}+\frac{1}{b+ac}+\frac{1}{c+ab}=\frac{27}{4}$.

4. Solve the following equations: 1) $\frac{4(x+5)}{x+6}+\frac{3(x+6)}{x+5}=7$

(2) $x+y+z=ax+by+cz=0;\ \frac{x}{b-c}+\frac{y}{a-c}+\frac{z}{a-b}=1.$

5 A mail coach runs between two places A and B and back again. A traveller who starts walking from A, 5 hours before the mail coach is overtaken by it half-way between A and B. He then doubles his rate of walking and meets the mail coach on his return journey 3 miles from B. The traveller then goes to B at the same rate and returns, and by the time he comes again midway between A and B, the mail coach reaches A. Find the distance between A and B and the rate at which the mail coach runs.

1879.

1. Simplify $\frac{(a+b)^2}{(x-a)(x+a+b)}-\frac{a+2b+x}{2(x-a)}+\frac{(a+b)x}{x^2+bx-a^2-ab}+\frac{1}{2}$.

2. Transform $(x^2+y^2+z^2+2xy)^2$ into the sum of two perfect squares.

3 Is $(a+b+c)\{a^2-(b+c)a+(b+c)^2\}-3bc(b+c)=(a+b+c)$ $\times\{b^2-(a+c)b+(a+c)^2\}-3ac(a+c)$ an Equation or an Identity?

4. Solve the equations:

(1) $\frac{x+7}{4}+\frac{2x-4}{7}=\frac{3x-12}{5}+3.$

(2) $x^2+3x-10 = ax^2+bx^2+cx^2+5(a+b+c)x.$

(3) $ax+by+cz = a+b;\ bx+cy+az=b+c;\ cx+ay+bz=c+a.$

5. Three equal vessels A, B, C are placed one above another, A being the highest. A is full, B is half-full, and C is empty. In the bottom of A is a hole which would empty it in 16 minutes, and in the bottom of B one which would empty it in 4 minutes. How long will it be before C is full?

1880.

1. Multiply $a^2+2b^2+9c^2-3ab+6ac-9bc$ by $a+2b-3c$ and divide the result by $a-b+3c$.

2. Simplify the fraction $\frac{(a^4-b^4)^2+2a^6b^2+5a^4b^4+2a^2b^6}{(a^2+ab+b^2)^2(a^2-ab+b^2)^2}$

3. Find the value of $\sqrt[3]{[\,(x+2)\sqrt{x-2}-2\ \{\sqrt[3]{11x^2-x+2\sqrt{(x-2)}}\}\,]}$ when $x=11$.

4. Find the square roots of–

(1) $x^4-4x^3+6x^2-4x+1.$ (2) $x^2+4x+10+\frac{12}{x}+\frac{9}{x^2}.$

5. Solve the equations:

(1) $\frac{x+3}{8}-\frac{x-3}{10}=\frac{x+5}{6}-\frac{x-7}{3}.$

(2) $\frac{1}{3}\left(\frac{x}{2}-2\right)-2(x-30)=\frac{1}{7}(x-6)-7.$

(3) $(a^2-b^2)x-(a^2-ab+c^2)y=a(a-2b)-\frac{bc^2}{a-b}$, $\frac{x}{a}+\frac{y}{b}=\frac{2a}{a^2-b^2}.$

6. A, B, C, D are four Railway Stations. From B to C is $2\frac{2}{3}$ miles more, and from C to D $5\frac{2}{7}$ miles less than from A to B. A train starts from A and travels at the rate of 14 miles an hour. At B an accident happens to the engine which causes a delay of 6 hours. After this the train proceeds to C at half speed. There another delay of $\frac{1}{2}$ an hour occurs, and then the train moves on to D at a speed further diminished by 1 mile. A man starts from A at the same time as the train, and travels straight across country to D, a distance of 58 miles. Including stoppages he averages 3 miles an hour and reaches D just with the train. What is the distance by Rail from A to D?

1881.

1. Shew that $x^3+6(y+z)x^2+12(y+z)^2x+8(y+z)^3=4(2y+3x+6z)y^2+(x+6y+2z)(x+2z)^2.$

2. Find the square root of $a+2\sqrt{2ab}+2b+4\sqrt{2ac}+8\sqrt{bc}+8c$.

3. Simplify the fraction: $\dfrac{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}{3(2a+b+c)(a+2b+c)(a+b+2c)}$.

4. Solve the equations—

(1) $\dfrac{(x-1)(x-2)(x-6)}{(x-3)^3}=1$. (2) $\dfrac{3}{2x}+\dfrac{2}{3y}=5$; $\dfrac{2}{x}+\dfrac{3}{y}=13$.

(2) $x+y-3z=-a$; $z+x-3y=-b$; $y+z-3x=-c$.

5. A steamer sailed from a certain port with 1st, 2nd and 3rd class emigrants, numbering in all 100. The fares of the three classes were in the proportion of 4 : 2 : 1, and the total amount received was £3,780. When she had completed two-thirds of her voyage, the steamer broke down, and a passing vessel was requested to take all her 3rd class and half of her 2nd class passengers for the remainder of the voyage, for a proportionate part of their fares, which would have amounted to £420. This was refused for want of accommodation, but an offer was made to take, on like conditions, one quarter both of the 1st and 2nd class passengers. This was accepted, and £240 paid for the service. How many passengers were there in each class? What were the respective fares?

1882.

1. If $a=4$, $b=3$, $c=2$, $d=1$, $e=0$, find the value of

$$3(b+d)\,\{6(a-d)^3+b(a-c)^2\}\;-\;(c+d)\,\{15(c-a)^3-(a+c)^2d\}$$
$$+(b+c)\,\{(b-3c)^3+(a-d)^3\}\;-(a+c)^2(b+c)^2de.$$

2. Find the G.C.M. of $3x^4-10x^3y+22x^2y^2-22xy^3+15y^4$ and $2x^4-7x^3y+16x^2y^2-17xy^3+12y^4$.

3 Simplify—

$$\frac{a^4+b^4+ab(a^2+b^2)}{(a+b)^2}-\frac{a^4+b^4-ab(a^2+b^2)}{(a-b)^2}\div\frac{12a^2b^2}{(a+b)^2-(a-b)^2}.$$

4. Find the square root of—

$$\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5.$$

5 Solve the following equations:—

(1) $r(q+x)-pr=t(q+x)-pt$. (2) $\dfrac{16x-27\frac{1}{5}}{3x-4}+\dfrac{77-x}{3(x-1)}=5+\dfrac{23}{x-1}$.

6. Two parties of workmen are placing sleepers for parallel lines of railway. The first set had placed 36 sleepers when the second began, and place 8 sleepers to the second set 7. The first, however, have to place 4 sleepers in the same space in which the second place 3. At what distance from the starting point will the one overtake the other, supposing there are 1764 of the more closely placed sleepers in a mile?

1883.

1. If $V=5a+4b-6c$, $X=-3a-9b+7c$, $Y=20a+7b-5c$, $Z=13a-5b+9c$, calculate the value of $V-(X+Y)+Z$.

2. Divide a by $a-2x$ to six terms.

3. Separate $12x^4+x^2y^2-y^4$ into three factors.

4. Reduce to their simplest form—

(a) $\left\{\dfrac{ap^2+aq^2+2bpq}{p^2+q^2}\right\}^2-\left\{\dfrac{bq^2+bp^2+2apq}{p^2+q^2}\right\}^2.$

(b) $\dfrac{xy+2x^2-3y^2+4yz+xz-z^2}{2x^2-9xz-5xy+4z^2-8yz-12y^2}.$

5. Find the square root of $\dfrac{4x^2}{9y^2}-\dfrac{x}{z}-\dfrac{16x^2}{15yz}+\dfrac{9y^2}{16z^2}+\dfrac{6xy}{5z^2}+\dfrac{16x^2}{25z^2}.$

6. Solve the following equations

(1) $x-\dfrac{x-2}{2}=5\frac{3}{4}-\dfrac{x+10}{5}+\dfrac{x-2}{4}$. (b) $ax+by=1=bx-\dfrac{b}{a}+ay-\dfrac{a}{b}$.

(c) (1) $x+y+z=1$. (2) $ax+by+cz=0$. (3) $a^2x+b^2y+c^2z=abc$.

7. A merchant engaged two writers, their pay being Rs. 60 for the first month, with a fixed monthly increase afterwards. They agreed to serve for one year, and each of them placed in the merchant's hands a deposit to be forfeited in proportion to the part of the year during which he might not serve. One remained at his post $7\frac{1}{2}$ months, and received for salary and portion of deposit returned Rs 537. The other remained 10 months, and received in like manner Rs. 728-5-4. What was the monthly increase? And what the amount of the deposit?

1884.

1. Find the value of $(ma-nb)(mb-nc)(mc-na)+(na-mb)(nb-mc)\times(nc-ma)$, when $a-b=0$.

2. Reduce to its simplest form: $\dfrac{a^3+11a+12}{a^5+11a^3-54}.$

3. Shew that $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}=m\dfrac{(a-b)(b-c)(c-a)}{(m+ab)(m+bc)(m+ca)}.$

4. Find the square root of $(a+4)^4+8a^2(a-4)^2-256a^2$.

5. Solve the equations—

(i) $\dfrac{3}{10x+9}+\dfrac{4}{45x+2}=\dfrac{7}{18x+5}$; (ii) $\dfrac{x}{a}+\dfrac{y}{b}=2$, $ax-by=a^2-b^2$.

6. A letter carrier has to go daily from P to Q in a prescribed time. If he goes a mile an hour faster than his ordinary rate, he arrives at Q half an hour before the time. But if he goes a mile an hour slower, he arrives three-quarters of an hour too late. Find his ordinary rate, and the distance from P to Q.

1885.

1. Subtract $b\,\{a-(b+c)\}$ from the sum of $a\,\{a-(c-b)\}$ and $c\,\{a-(b-c)\}$, and obtain the continued product of $(a+b+c)(a+b-c)\times(a+c-b)(b+c-a)$.

2. Reduce $\dfrac{x^4-4x+3}{2x^5-11x^2-9}$ to its lowest terms.

3. Find the square root of $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$.

4. Solve the equations—

$$(1)\ \frac{x}{x-2}+\frac{9-x}{7-x}=\frac{x+1}{x-1}+\frac{8-x}{6-x}.\qquad(2)\ \frac{m}{x}+\frac{n}{y}=a,\ \frac{n}{x}+\frac{m}{y}=b.$$

5. A set of bearers on a journey perform one-third of the distance at a certain rate and then halt one hour to take their food. The remainder of the journey is accomplished at only two-thirds of the former rate, and the bearers reach their destination in 7 hours after first starting. Had they travelled at the former rate $4\frac{1}{2}$ miles further than they did before halting, they might have halted $22\frac{1}{2}$ minutes longer and yet reached the end of their journey in the same time. Find the length of the journey.

1886.

1. Simplify $24\,\{x-\frac{1}{2}(x-3)\}\,\{x-\frac{2}{3}(x+2)\}\,\{x-\frac{3}{4}(x-1\frac{1}{3})\}$ and subtract the result from $(x+2)(x-3)(x+4)$.

2. Reduce to its simplest form $\dfrac{8xy-x^2-4y^2}{7x^2y^2-2x^4+4y^4}$.

3. Find the square root of $\dfrac{x^4}{4}-\dfrac{2x^3}{3}-\dfrac{11x^2}{36}+x+\dfrac{9}{16}$.

4. Solve the following equations :—

$$(1)\ \frac{(x-a)(x+b)}{x-a+b}=\frac{x(x-c)-b(x+c)}{x-b-c}.$$

$$(2)\ \tfrac{1}{2}x+\tfrac{2}{3}y+\tfrac{1}{6}=0;\ \tfrac{1}{2}y+\tfrac{2}{3}x-\tfrac{5}{72}=0.$$

5. Two men A and B are employed on a piece of work which has to be finished in 14 days. In 3 days they do one-fifth of the work, and then A's place is taken by C. B and C work for one day and do one-twentieth of

the whole work, and then B's place is taken by A. A and C finish the work a day before the appointed time. Find the time in which the work could have been done (1) by each working separately, (2) by all working together.

1887.

1. Prove that $(x-y)(x+1)(y+1)-x(y+1)^2+y(x+1)^2=(x-y)\times(x+y+2xy)$.

2. Shew that the product of any two expressions is equal to the product of their H.C.F. (*i.e.*, G.C.M.) and L.C M.

Find the H.C.F. of the expressions—

$x^4-6x^3+7x^2+6x+8$ and $2x^3-11x^2+11x+4$.

3. Simplify $\dfrac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}\div\dfrac{x^4-y^4}{2xy(x-y)}$;

and prove that $\dfrac{(a+1)^2}{(a-b)(a-c)}+\dfrac{(b+1)^2}{(b-c)(b-a)}+\dfrac{(c+1)^2}{(c-a)(c-b)}=1$.

4. Solve the equations—

(1) $\dfrac{1}{x-1}-\dfrac{1}{x}-\dfrac{1}{x+3}+\dfrac{1}{x+4}=0$. (2) $\dfrac{b}{x}+\dfrac{a+c}{y}=m$; $\dfrac{a-c}{x}+\dfrac{b}{y}=n$.

5. A number consists of two digits. When the number is divided by the sum of its digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number.

1888.

1. (1) Divide $a^3+8b^3+27c^3-18abc$ by $a^2+4b^2+9c^2-6bc-3ca-2ab$.

(2) Resolve into three factors $(x+1)(x+3)(x+5)(x+7)+15$.

(3) Shew that $4(a^2+ab+b^2)^3-(a-b)^2(a+2b)^2(2a+b)^2$
$=27a^2b^2(a+b)^2$.

2. Find the G.C.M. and L.C.M. of—

$3x^4+17x^3+27x^2+7x-6$ and $6x^4+7x^3-27x^2+17x-3$.

3. If $y=\dfrac{1+x}{1-x}$ prove that $\left(x-\dfrac{1}{x}\right)\left(y-\dfrac{1}{y}\right)=4\dfrac{xy+1}{x-y}$.

4. Solve the equations—

(1) $\dfrac{a+c}{x-2b}-\dfrac{b+c}{x-2a}=\dfrac{a-c}{x+2b}-\dfrac{b-c}{x+2a}$. (2) $2x-3y+z+1=0$;

$5x-3z=6$, $3x+2y=4$.

5. In a quarter-of-a-mile race A gives B a start of 22 yards, and beats him by 2 seconds; and in a 300 yards' race, he gives him a start of 2 seconds, and beats him by $10\frac{1}{2}$ yards. Find the rates of each.

1889.

1. Simplify the expression-

(1) $x-[a-\{2a-(3a-4a-x)\}]$. (2) $\{(x^{a+b-c}\times x^{a-b+c})^b\}$.

(3) $(a+b)^m\times(a-b)^m\times(a^2+b^2)^m$.

2. Arrange the expression $x(p+x)\{p^2+q^2-x(p-x)\}-(p^2+qx)\times(2x^2-qx+q^2)$ in powers of x; and divide it by $x^2+(p-q)x-p^2$.

3. (1) If $x^2=ab+bc+ca$, shew that $(a^2+x^2)(b^2+x^2)(c^2+x^2)$ is a perfect square.

(2) Extract the square root of $1-xy-\frac{15}{4}x^2y^2+2x^3y^3+4x^4y^4$.

4. (1) Find the G.C.M. of $7x^3-19x^2+17x-5$ and $2x^4-x^3-9x^2+13x-5$.

(2) Simplify the expression—

$$\frac{b^2+c^2-2a^2}{(a-b)(a-c)}+\frac{c^2+a^2-2b^2}{(b-c)(b-a)}+\frac{a^2+b^2-2c^2}{(c-a)(c-b)}.$$

5. Solve the equations—

(1) $\dfrac{(x+1)(x+9)}{(x+2)(x+4)}=\dfrac{(x+6)(x+10)}{(x+5)(x+7)}$. (2) $x+\sqrt{(x^2+9)}=9$.

(3) $x-2y=5$, $3y+4z=6$, $5z+6x=21$.

6. A person bought 166 mangoes for ten rupees; some he bought at the rate of 18 per rupee, and the rest at 15 per rupee. How many did he buy of each sort?

1890.

1 Simplify the following expressions, arranging the last in ascending powers of x:—

(1) $\{m-n-(3x-2y)\}-[3m+2n-\{x-y+(m+2n)-(2y-x)\}]$.

(2) $\left(\dfrac{x^a}{x^b}\right)^{a+b}\div\left(\dfrac{x^{a+b}}{x^{a-b}}\right)^{\frac{a^2}{b}}$.

(3) $(x^2-xy+y^2)(x^2-2xy+y^2)(x^2+xy+y^2)(x^2+2xy+y^2)$.

2. Find the G.C.M. of $6x^3+7x^2-9x+2$, and $8x^4+6x^3-15x^2+9x-2$ and write down the L.C.M. in factors.

3. Extract the square root of $x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2$.

4. Simplify (1) $\dfrac{a^2-(b-c)^2}{(a-b)(a-c)}+\dfrac{b^2-(c-a)^2}{(b-c)(b-a)}+\dfrac{c^2-(a-b)^2}{(c-a)(c-b)}$.

(2) $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}$.

5. Solve the following equations :—

(1) $\dfrac{x-1}{x-2}+\dfrac{x-4}{x-5}=\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$. (2) $x-y+z=1$; $x-2y+4z=8$; $x-3y+9z=27$.

6. A sum of Rs. 63 4 as. was paid in rupees and two-anna pieces. The total number of coins being 100, how many of each kind were used ?

1891.

1. Simplify the expression—

$$7(a-3b+c)-[4(2b+4c)(6c-3b)-3(a-4b)(a+3b)+\{(5a-4b)3c+4+\overline{a-47b+c}\}+7].$$

2. Divide $\frac{1}{2}-x$ by $\frac{1}{4}-x+x^2$ to *five* terms.

3. (*a*) Find the G.C.M. of $x^4-8x^3+28x^2-53x+42$ and $x^4+6x^3-42x^2+129x-154$.

(*b*) Shew that $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}=a+b+c$.

4. Simplify $\dfrac{3x-12}{x^2-5x+6}+\dfrac{5x-3}{x^2-2x-3}+\dfrac{x+15}{x^2-5x-6}$.

5. Solve the equations—

(*a*) $\dfrac{3x-2}{5}+\dfrac{4x-1}{7}-\dfrac{10x}{9}=5(x-9)+3-\dfrac{x}{3}$.

(*b*) $49x-57y=172$; $57x-49y=252$.

(*c*) $\dfrac{y}{a}+\dfrac{x}{b-a}=5m$; $\dfrac{x}{b}+\dfrac{y}{a-b}=7m$.

6. A composition of copper and tin containing 140 cubic inches weighs 42 lbs. 3 oz. How many ounces of each are there if a cubic inch of copper weighs 5¼ oz. and a cubic inch of tin 4¼ oz. ?

1892.

1. If a, b, c be three quantities whose sum is zero, shew that $a^4+b^4+c^4=2(a^2b^2+b^2c^2+a^2c^2)$.

2. Break into factors $a^3b+b^3c+c^3a-(ab^3+bc^3+ca^3)$

3. Simplify—

$$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}.$$

4. Extract the square root of—

$$\frac{4x^2}{9a^2}+\frac{4x^2}{b^2}+\frac{9a^2}{4x^2}+\frac{9a^2x^2}{b^4}+\frac{9a^2}{b^2}+2.$$

5. Prove the rule for finding the G.C.M. of two quantities, and find that of $7x^4-2x^2-9x-2$ and $5x^3-6x^2-6x-11$.

6. Solve (1) $\begin{cases} 2x-\dfrac{y+3}{4}=7+\dfrac{3y-2x}{5} \\ 4y+\dfrac{x-2}{3}=26\frac{1}{2}-\dfrac{2y+1}{2}. \end{cases}$

(2) $\dfrac{x+16}{5}+\dfrac{11}{x}=\dfrac{4x-17\frac{1}{3}}{3}.$

7. In a half-mile race, A gives B 22 yards' start and wins by 6 seconds. In a three-quarter mile race, he gives him 20 seconds' start but is beaten by 29 yards 1 foot. In what time can each of them run a mile?

1893.

1. (1) From $a(b+c)^2+b(a+c)^2+c(a+b)^2$, subtract $(a+b)(a-c)\times(b-c)+(b+c)(a-b)(a-c)-(a+c)(a-b)(b-c)$.

(2) Shew that $(5x^2-6x+7)^2-(5x^2+16x+3)^2$ is divisible by x^2+x+1, and find the quotient.

(3) Divide $x^4+5ax^3+(25a-b-29)x^2-5(4a+b-4)x+4b$ by x^2+5x-4.

2. (1) Find the four factors of—

$$(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4.$$

(2) Simplify $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}\div\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}{\dfrac{1}{b}-\dfrac{1}{a}}.$

3. Find the G.C.M. of—
$6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$.

4. Extract the square root of—
$x^{4m+2}+6x^{3m+1}-10x^{2m+1}y^{m-2}+9x^{2m}-30x^my^{m-2}+25y^{2m-4}$.

5. Solve the equations—

(1) $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}\\ \frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}\end{aligned}\right\}.$ (2) $\frac{3}{5-x}+\frac{2}{4-x}=\frac{8}{x+2}.$

6. A boy bought a number of oranges for 2 Rupees. Had he bought 8 more for the same money, he would have paid 4 pies less for each. How many did he buy?

1894.

1. Simplify the expressions—

(1) $\left[\sqrt{\frac{x^2}{y^4}}\times\sqrt{\frac{y^3}{x^5}}\right]^{12}\times x^{22}.$

(2) $2(z^3+x^3)-[(x+y)(xy-x^2-y^2)-\{2(x+y+z)$
$\times(yz+xz+xy-x^2-y^2-z^2)-(x-y)(x^2+xy+y^2)\}].$

2. Divide the product of—

$ab(x^2+1)+(a^2+b^2)x$ and x^3+1 by that of $x+1$ and $ax+b$.

3. Extract the square root of—

$$x^4(x-a)^2+4a^2x^4+a^4(2x+a)-2a^3x^2(x+a).$$

4. Find the L.C.M. of $(x-a)$, x^2-a^2, x^3-a^3, $(x^3+a^3)^2$; and the H.C.F. of x^5+11x^3-54 and $2x^5-11x^2-9$.

5. Express—

$$\frac{(a-b)(b-c)}{(c-d)(d-a)}-\frac{(b-c)(c-d)}{(d-a)(a-b)}+\frac{(c-d)(d-a)}{(a-b)(b-c)}-\frac{(d-a)(a-b)}{(b-c)(c-d)}$$

as a fraction whose numerator and denominator consist of four factors each.

6. Solve the equations—

(1) $\frac{(x+1)(x+2)}{x+3}=\frac{(x+4)(x+5)}{x+9}.$

(2) $3x+4y+2z=19;$
$7x-3y=15;$
$7z-4y=-1.$

(3) $\sqrt{x^2+3x+15}+2x=9.$

7. If 3 be added to the numerator and denominator of a certain fraction, the fraction becomes $\frac{2}{3}$; if 5 be subtracted from the numerator and denominator, it becomes $\frac{1}{2}$. Find the fraction.

8. A party of travellers coming to an hotel find that there are a too few bed-rooms for each to have one. If they sleep two in a room there are b empty rooms. How many rooms are left empty if they sleep three in a room?

1895.

1. Shew that $(4x^2-8x-1)^2-(2x^2-5x+7)^2$ is divisible by $2x^2-3x-8$; and express the quotient as the product of two factors.

2. (1) Simplify $\left\{1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right\}\left\{1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right\}$.

(2) Prove that—

$$\frac{a(x-b)(x-c)}{bc(a-b)(a-c)}+\frac{b(x-c)(x-a)}{ac(b-c)(b-a)}+\frac{c(x-a)(x-b)}{ab(c-a)(c-b)}=\frac{x^3}{abc}.$$

3. Find the H.C.F. of $x^3+2x^2-3x+20$ and $3x^4-34x^2+51x-20$.

4. Extract the Square root of—

$$x^6+2x^4+4x^3+x^2+4x-\frac{4}{x^2}-\frac{8}{x}+\frac{4}{x^6}.$$

5. Solve the equations—

$$(1)\ \frac{1}{x+1}+\frac{2}{2x+3}=\frac{6}{3x+5}.$$

$$(2)\ \left.\begin{array}{l}\dfrac{x-y}{a}+\dfrac{x+y}{b}=c\\[2mm]\dfrac{x-y}{b}-\dfrac{x+y}{a}=c\end{array}\right\}. \qquad (3)\ 141x^2-88x-45=0$$

6. The gross income of a certain person was Rs. 4 more in the second of two particular years than in the first, but as he paid income-tax at the rate of 4 pies in the rupee in the first year and at the rate of five pies in the rupee in the second year, his net income in the second year was Rs. $6\frac{1}{4}$ less than his net income in the first. What was his gross income in each year?

1896.

1. Simplify—

$$(1)\ \frac{1}{x^2+3x+2}+\frac{x-1}{2x^2+5x+2}-\frac{x}{2x^2+3x+1};$$

(2) Prove that
$$\frac{(a+2b-3c)^2}{(b+2c-3a)(c+2a-3b)}+\frac{(b+2c-3a)^2}{(c+2a-3b)(a+2b-3c)}+\frac{(c+2a-3b)}{(c+2b-3c)(b+2c-3a)}=3.$$

2. Find the highest common factor of—

$6x^4+2x^3+19x^2+8x+21$ and $4x^4-2x^3+10x^2+x-15$.

3. Find the lowest common multiple of—

$$x^3+a^3,\ x^3-a^3,\ x^4+a^2x^2+a^4, \text{ and } x^2-ax+a^2.$$

4. Extract the Square root of—

$$4x^2+4x-11-\frac{10}{x}+\frac{7}{x^2}+\frac{6}{x^3}+\frac{1}{x^4}.$$

5. Solve the equations—

(1) $\dfrac{(x-3)(x-4)}{x-7}=\dfrac{(x+1)(x+3)}{x+4}$.

(2) $\dfrac{1}{x}+\dfrac{3}{y}-\dfrac{2}{z}=6,\ \dfrac{3}{x}+\dfrac{1}{z}=5,\ \dfrac{2}{y}+\dfrac{5}{z}=16.$

(3) $129x^2-34x-80=0.$

6. One person starts from a place A to walk to a place B and back again at the same time as another person starts from B to walk to A and back again. They meet first at a distance of two miles from A and afterwards at a distance of 4 miles from A. Find the distance between A and B.

1897.

1. Perform the multiplications—

(1) $(3a^4-4a^3x-5x^4)(3a^4+4ax^3+5x^4)$.

(2) $(x+a)^3(x-a)^5$.

2. Divide (1) $x^5-y^5+\dfrac{y^{10}}{x^5}$ by $x-y+\dfrac{y^2}{x}$.

(2) $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3$ by $(b-c)\times(c-a)(a-b)$.

3. Resolve into factors—

(1) $16x^8-1$.

(2) $4(ac+bd)^2-(a^2-b^2+c^2-d^2)^2$.

(3) $(x-1)(x-2)-2(y-1)(x-2)+(y-1)(y-2)$.

4. Simplify (1) $\dfrac{(b+c-2a)^3-(c+a-2b)^3}{(c+a-2b)^3-(a+b-2c)^3}$.

(2) $\dfrac{a+b}{a-b+\dfrac{b^2}{a+b}}-\dfrac{a-b}{a+b+\dfrac{b^2}{a-b}}$.

7. If 3 .

ction, the

d denominat $+\dfrac{1}{y}+\dfrac{1}{z}=0$, prove that $\tfrac{1}{2}\left(\dfrac{y^2z^2}{x^2}+\dfrac{z^2x^2}{y^2}+\dfrac{x^2y^2}{z^2}\right)=(x+y+z)^2$.

6. Solve the equations :—

(1) $\frac{1}{3}(x-4)+\frac{3}{7}(2x-7)-\frac{7}{9}(1+5x)=4(1-x)$.

(2) $\dfrac{4}{2x+3}-\dfrac{18}{7x+12}=5$.

(2) $3x-4y+5z+26=0=3y-4z+5x-13=3z-4x+5y-5$.

7. When the price of sugar rises 50 per cent. and the price of tea 10 per cent., the increase in the cost of 3 lbs. of tea and 4 lbs. of sugar, which together originally cost Rs. 3-8-0 is 12 as. What is the original price of tea ?

1898.

1. D vide (a) $(x^2-1)^4-3(x^2-1)^2+1$ by x^4-3x^2+1;
(b) $a^2(1-x)+ab(a-b)(x+y)+b^2(1+y)$ by $a(1-x)+b(1+y)$.

2. Resolve into factors :—

(1) $(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2$; (2) $a^3(a+1)+b^3(b+1)-ab(a-b)^2$.

3. Find the H.C.F. of $3x^4-2x^3+2x^2+8$ and $x^3-7x^2+12x-10$.

4. (a) Simplify $\dfrac{1}{x^3-x^2+x-1}+\dfrac{3}{2x^2-x-1}+\dfrac{1-3x}{2x^3+x^2+2x+1}$.

(b) Prove that $\dfrac{(a-5b)(3a+b)^3+(5a-b)(a+3b)^3}{a+b}=32\,(a-b)^3$.

5. Extract the square root of $b^2(a+4b)^2+3(3a^2-2ab+b^2)(a^2+3b^2)$.

6. Solve the equations :—

(1) $\dfrac{2x-7}{2x-8}-\dfrac{3x+8}{3x+9}=\dfrac{x-3}{x^2-x-12}$; (2) $x+\dfrac{ay}{a-b}=b=\dfrac{ax}{a+b}+y$;

(3) $\dfrac{x-3}{x-2}+\dfrac{x-4}{x+1}+\dfrac{1}{4}=0$.

7. The length and breadth of a room are such that if the former were increased and the latter diminished by 3 yards, the area of the room would be diminished by 18 square yards, while if both were increased by 3 yards, the area would be increased by 60 square yards. Find the length and breadth of the room.

1899.

1. Multiply $x^2+(3a-2b)x-6ab$ by $x^2+(2a-3b)x-6ab$; and divide $(a+1)^2x^3+(a+1)x^2+a^2(a-1)x-a^5$ by $(a+1)x-a^2$.

2. Resolve into factors :—(1) $(ab+1)^4-4ab(ab+1)^2-(a^2-b^2)^2$;
(2) $(a^2-b^2)(a+b)+(b^2-c^2)(b+c)+(c^2-a^2)(c+a)$.

3. Reduce to its lowest terms $\dfrac{3x^3-23x^2+43x-8}{x^4-5x^3-6x^2+35x-7}$.

4. Show that $\dfrac{(a+3x+2b)(a+x)^3-(2a+3x+b)(b+x)^3}{(a+2x+b)^3}=a-b$.

5. Extract the square root of—

$$x^6+\frac{1}{x^6}+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20$$

and shew that $\dfrac{1}{x^3}+\dfrac{1}{y^3}+\dfrac{1}{z^3}=\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)^3+\dfrac{3}{xyz}$, if $x+y+z=0$.

6. Solve the equations :—

(1) $\dfrac{x+5}{2x-1}-\dfrac{3(5x+1)}{5x+4}=\dfrac{4}{2x-1}-2\tfrac{1}{3}$;

(2) $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=\dfrac{2x+13}{x+1}$.

(3) $3x-4y-6z+16=0=4x-y-z-5=x-3y-4z+12$.

7. A walks half a mile per hour faster than B and three-quarters of a mile per hour faster than C. To walk a certain distance C takes three-quarters of an hour more than B, and two hours more than A. Find the rates of walking of A, B and C.

March 1900.

1. (1) Multiply $(x^2+2ax+4a^2)^2$ by $(x-2a)^3$.

(2) Divide $6x^6-19x^5+6x^3-3x+2$ by $3x^2-2x+1$.

2. Resolve each of the following expressions into four factors :—

(1) $(b+c)^2-2(b^2+c^2)a^2+(b-c)^2a^4$.

(2) $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.

3. Find the highest common factor of—

$2x^4+13x^3-4x^2+6x+1$ and $x^4+7x^3-2x^2-21x-3$.

4. Simplify (1) $\dfrac{1}{x^4+2x^3}+\dfrac{1}{x^4-2x^3}+\dfrac{2}{x^4+4x^2}$;

(2) $\left\{\dfrac{x-a}{(x+a)^2}+\dfrac{x+a}{(x-a)^2}\right\}\div\left\{\dfrac{1}{(a+x)^2}+\dfrac{1}{(a^2-x^2)}+\dfrac{1}{(a-x)^2}\right\}$.

5. Extract the square root of

$(a^2+3b^2)^2+10ab(a+b)(a-3b)+33a^2b^2$.

6. Solve the equations.

(1) $\dfrac{2x-3}{5}+\dfrac{9}{10}(3x+8)=5x+\tfrac{1}{3}(4x-19)$.

(2) $\dfrac{3x-1}{3x-9}+\dfrac{x+1}{x-7}=\dfrac{x-10}{x^2-10x+21}$;

(3) $x+2y+3z=\dfrac{19}{6}$; $2x+3y+z=2$; $3x-4y-7z=\dfrac{1}{6}$.

7. A person has a certain amount of money to divide among a number of people. If he gives 2 rupees to each he will have 20 rupees left over, but there are 10 people too many for him to give Rs. 2-8 to each. Find the amount he has to give away, and the number of people.

December 1900.

1. (1) Find the value of $\frac{1+x^3}{(1+x)^2}+\frac{1-x^3}{(1-x)^2}$, when $x^2=\frac{m-2}{m+4}$.

(2) Multiply the square of $\frac{x^2}{2}-\frac{x}{3}+\frac{1}{4}$ by $\frac{x^2}{4}+\frac{x}{3}-\frac{1}{2}$

2. Divide $(x^2-a^2)(x+a)b+(x^2-b^2)(x-b)a-(a^2-b^2)(a-b)x$ by $x-a$ and the quotient by $a+b$; and hence resolve the expression into four factors.

3. Find the H.C.F. of x^4-41x^2+16 and $x^4-7x^3+28x-16$.

4. (1) Simplify the expression $\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}$.

(2) Show that the expression

$$4ab(a-b)^2\,(a+b+\sqrt{ab})^2+\{(a-b)^2(a+b)-4ab\sqrt{ab}\}^2$$ is

a perfect square.

5. If $a+b+c=0$, show that $a^4+b^4+c^4+(bc+ca+ab)(a^2+b^2+c^2)=0$

6. Solve the equations :—

(1) $\frac{75-x}{3(x+1)}+\frac{80x+21}{5(3x+2)}=\frac{23}{x+1}+5$;

(2) $\frac{3}{5-x}+\frac{2}{4-x}=\frac{8}{x+2}$;

(3) $x+2y=3z+1$,
$y+2z=2(x-1)$,
$z+2x=5y+1$.

7. A grocer mixed 28 lbs. of tea worth 14as. a pound with tea worth Re. 1 2as. a pound and tea worth Re. 1 8as. a pound. If he had 100 lbs. of the mixture worth Re. 1 4as. a pound, how many pounds of the best quality did he take?

MADRAS UPPER SECONDARY EXAMINATION PAPERS.

May 1890.

1. Shew that the expressions $a-(b+c)-(d-e)$ and $a-b-c-d+e$ are equivalent.

2. (*a*) Prove that—

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)+4.$$

(*b*) Find the value of $x^4-2x^3y+2xy^3-y^4$ when $x=a+b$, and $y=a-b$.

(*c*) Shew that, for certain values of n, the expression $(ma+b)^{2n}+a^{\frac{n}{2}}-(a+mb)^{2n}-b^{\frac{n}{2}}$ is divisible by $a-b$ or $a+b$.

3. (*a*) Prove that if an algebraical expression be the G.C.M. of two other expressions, it is also the G.C.M. of their sum and difference.

(*b*) Determine the H.C.F. of the expressions $(a^2-2a)x^2+2(2a-1x)-a^2+1$ and $(a^2-a-2)x^2+(4a+1)x-a^2-a$.

4. Solve the equations: (1) $\sqrt{x}+\sqrt{(a+x)}=2a\left(a+x\right)^{-\frac{1}{2}}$.

(2) $ax\left(\dfrac{1}{a-b}-\dfrac{1}{a+b}\right)+cy\left(\dfrac{1}{b-a}-\dfrac{1}{b+a}\right)=\dfrac{2a}{a+b}$; $bx+cy=a+b$.

5. AB is a railway 210 miles long. Two trains start from A at 6 A.M. and 7-30 A.M. respectively, and a third train leaves B at 8 A.M. Supposing the speed of the trains is 25, 20 and 30 miles an hour respectively, at what distance from A and at what hour will the first train be equidistant from the two others?

May 1891.

1. Divide $1+x^3+y^3-3xy$ by $1+x+y$.

2. (*a*) Resolve into factors—

(1) $(x-y)^3+(y-z)^3+(z-x)^3$; (2) $9x^4-10x^2y^2+y^4$.

(*b*) Find the value of $a^3-4a^2b-4ab^2+b^3$ when $a+b=4$ and $ab=2$.

3. (*a*) Enunciate and prove the rule for finding the L.C.M. of two algebraical expressions.

(*b*) Express in the form of two factors the L.C.M. of—

$$a^3x^4+a^2bx^3+ac(a-c)x^2-bc^2x-c^3 \text{ and}$$
$$a^3x^4-a^2bx^3-ac(a+c)x^2+bc^2x+c^3.$$

4. Solve the equations—

(1) $\dfrac{a+b+bx}{a+b}-\dfrac{bx-a+b}{a-b}=\dfrac{x^2+2bx-a^2+b^2}{x-a+b}+2.$

(2) $\dfrac{x+y}{a+b}+\dfrac{x-y}{a-b}=\dfrac{1}{a^2-b^2}$; $\dfrac{a+b}{x+y}-\dfrac{a-b}{x-y}=\dfrac{1}{x^2-y^2}$.

5. Starting from a certain time between four and five o'clock, the minute hand of a clock takes half as long to reach a position 5 minutes behind the hour hand as it does to reach a position 5 minutes in advance of it. What is the time of starting?

APPENDIX IV.

Solutions and Hints to the Madras Matriculation Papers from 1892—1901.

N.B.—*References are to the articles in the body of the book.*

For the Papers see Appendix III, pp. 384 to 391.

1892.

1. $a+b+c=0$. $\therefore\ a^2+b^2+c^2=-2(ab+ac+bc)$

$\therefore\ a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)=4(ab+ac+bc)^2$
$=4(a^2b^2+b^2c^2+c^2a^2)$ [art 79]

$\therefore\ a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$.

2. $a^3b+b^3c+c^3a-(ab^3+bc^3+ca^3)$

$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$ [See Art. 76].

$=(b-c)(c-a)(b-a)(a+b+c)$

3. Numerator $=(y^2-z^2)(y+z)^2+(z^2-x^2)(z+x)^2+(x^2-y^2)(x+y)^2$

$=(y^2-z^2)(y^2+z^2)+(z^2-x^2)(z^2+x^2)+(x^2-y^2)(x^2+y^2)$
$+2\{yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)\}$
$=2\{yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)\}$ ...(A).

Denominator $=(y^2-z^2)(y-z)^2+(z^2-x^2)(z-x)^2+(x^2-y^2)(x-y)^2$

$=(y^2-z^2)(y^2+z^2)+(z^2-x^2)(z^2+x^2)+(x^2-y^2)(x^2+y^2)$
$-2\{yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)\}$
$=-2\{yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)\}$ ...(B)

$\therefore$ The given fraction $=\dfrac{A}{B}=-1$.

4. $\frac{4x^2}{9a^2}+\frac{4x^2}{b^2}+\frac{9a^2}{4x^2}+\frac{9a^2x^2}{b^4}+\frac{9a^2}{b^2}+2.$

$$=\left(\frac{4x^2}{9a^2}+2+\frac{9a^2}{4x^2}\right)+\frac{9a^2x^2}{b^4}+\frac{4x^2}{b^2}+\frac{9a^2}{b^2}.$$

$$=\left(\frac{2x}{3a}+\frac{3a}{2x}\right)^2+\left(\frac{3ax}{b^2}\right)^2+\frac{6ax}{b^2}\left(\frac{2x}{3a}+\frac{3a}{2x}\right)$$

$$=\left(\frac{2x}{3a}+\frac{3a}{2x}+\frac{3ax}{b^2}\right)^2. \quad \therefore \text{ square root is } \frac{2x}{3a}+\frac{3a}{2x}+\frac{3ax}{b^2}.$$

5. See Art 85

We shall find the G. C. M by the *method of detatched co-efficients.*

$$5-6-6-11)\ 7+0-2-9-2\ (7+42$$
$$5$$
$$35+0-10-45-10$$
$$35-42-42-77$$
$$42+32+32-10$$
$$5$$
$$210+160+160-50$$
$$210-252-252-462$$
$$412+412+412+412$$
$$1+1+1$$
$$1+1+1)\ 5-6-6-11\ (5-11$$
$$5+5+5$$
$$-11-11-11$$
$$-11-11-11$$

$\therefore$ The G. C. M is $1+1+1$ i.e., x^2+x+1

6. (1) $2x-\frac{y+3}{4}=7+\frac{3y-2x}{5}$

$\therefore\ 40x-5y-15=140+12y-8x$

$\therefore\ 48x-17y=155 \quad . \quad . \quad . \quad (A)$

Again, $4y+\frac{x-2}{3}=26\frac{1}{2}-\frac{2y+1}{2}$

$\therefore\ 24y+2x-4=159-6y-3$

$\therefore\ 2x+30y=160 \quad \therefore\ x+15y=80 \quad . \quad (B).$

From (A) and (B), $x=5$; $y=5$.

(2) $\frac{x+16}{5}+\frac{11}{x}=\frac{4x-17\frac{1}{3}}{3}$.

$\therefore\ 3x\ (x+16)+165=5x\ (4x-17\frac{1}{3})$

$\therefore\ 17x^2-134\frac{2}{3}x-165=0$

$\therefore\ 51x^2-404x-495=0$

$\therefore\ (51x+55)\ (x-9)=0$

$\therefore\ x=9$ or $-1\frac{4}{51}$.

7. Let $x=$ A's rate *in yards per second*,
And $y=$ B's rate *in yards per second*,

By the question, $\frac{880}{x}=\frac{880-22}{y}-6$. (1)

and $\frac{1320}{y}=20+\frac{1320-29\frac{1}{3}}{x}$. . . (2)

From (1),

$\therefore \frac{858}{y}-\frac{880}{x}=6 \qquad \therefore \frac{429}{y}-\frac{440}{x}=3.$

From (2), $\frac{1320}{y}-\frac{1290\frac{2}{3}}{x}=20$

$\therefore \frac{440}{y}-\frac{430\frac{2}{9}}{x}=6\frac{2}{3}$

$\therefore x=\frac{88}{15}$ and $y=\frac{11}{2}$

A can run a mile in $1760\div\frac{88}{15}$ or 5 minutes and B can run a mile in $1760\div\frac{11}{2}$ or $5\frac{1}{3}$ minutes.

1893.

1. (2) $(5x^2-6x+7)^2-(5x^2+16x+3)^2$.
$=(10x^2+10x+10)(-22x+4)$. $\because a^2-b^2=(a+b)(a-b)$.

2. (1) $(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4$. $\because (a+b)^2+(a-b)^2=2(a^2+b^2)$.

$$=(1+y)^2-x^2\{(1+y)^2+(1-y)^2\}+(1-y)^2x^4.$$
$$=(1+y)^2\{1-x^2\}-(1-y)^2x^2\{1-x^2\}.$$
$$=(1-x^2)\{(1+y)^2-(1-y)^2x^2\}.$$
$$=(1+x)(1-x)(1+y+x-xy)(1+y-x+xy).$$

(2) $$\left(\frac{a^6-b^6}{a^3b^3}\times\frac{ab}{a^2-b^2}\times\frac{ab}{a^2+b^2-ab}\right)\div\left(\frac{a^2+b^2+ab}{a^2b^2}\times\frac{ab}{a-b}\right).$$

$$=\frac{a^2+ab+b^2}{ab}\div\frac{a^2+b^2+ab}{ab(a-b)}=a-b.$$

3 $6x^5-4x^4-11x^3-3x^2-3x-1.$

$$=6x^4(x+1)-10x^3(x+1)-x^2(x+1)-2x(x+1)-(x+1)$$

(See Art 71).

$$=(x+1)(6x^4-10x^3-x^2-2x-1).$$

Since $x+1$ is *not* a factor of the other expression, the G. C. M. of the given expressions is the G. C M. of $6x^4-10x^3-x^2-2x-1$ and $4x^4+2x^3-18x^2+3a-5$ which can be found by the ordinary method

4. $x^{4m+2}+6x^{3m+1}-10x^{2m+1}y^{n-2}+9x^{2m}-30x^my^{m-2}+25y^{2m-4}$

$$\begin{array}{l|l|l}
 & x^{4m+2} & \\
\hline
2x^{2m+1}+3x^m & 6x^{3m+1}-10x^{2m+1}y^{m-2}+9x^{2m} & x^{2m+1}+3x^m-5y^{m-2} \\
 & 6x^{3m+1}+9x^{2m} & \\
\hline
2x^{2m+1}+6x^m-5y^{m-2} & -10x^{2m+1}y^{m-2}-30x^my^{m-2}+25y^{2m-4} & \\
 & -10x^{2m+1}y^{m-2}-30x^my^{m-2}+25y^{2m-4} & \\
\end{array}$$

∴ Sq. root is $x^{2m+1}+3x^m-5y^{m-2}$.

5. (1) $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$ and $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$.

$$\therefore\ x\left(\frac{1}{a}+\frac{1}{c}\right)+\frac{y}{b}=1\text{.....................(A).}$$

and $$x\left(\frac{1}{b}\right)+y\left(\frac{1}{a}-\frac{1}{c}\right)=1\text{...........(B).}$$

From (A), $$x\left(\frac{1}{a}+\frac{1}{c}\right)\left(\frac{1}{a}-\frac{1}{c}\right)+\frac{y}{b}\left(\frac{1}{a}-\frac{1}{c}\right)=\frac{1}{a}-\frac{1}{c}.$$

From (B), $x\dfrac{1}{b^2} \qquad\qquad + y\left(\dfrac{1}{a}-\dfrac{1}{c}\right)\dfrac{1}{b}=\dfrac{1}{b}.$

$\therefore\ x\left(\dfrac{1}{a^2}-\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)=\dfrac{1}{a}-\dfrac{1}{c}-\dfrac{1}{b}.$

$\therefore\ x=\left(\dfrac{1}{a}-\dfrac{1}{c}-\dfrac{1}{b}\right)\div\left(\dfrac{1}{a^2}-\dfrac{1}{c^2}-\dfrac{1}{b^2}\right).$

Similarly y can be found.

(2) $\dfrac{3}{5-x}+\dfrac{2}{4-x}=\dfrac{8}{x+2}.$

$\therefore\ 3(4-x)(x+2)+2(5-x)(x+2)=8(5-x)(4-x).$

$\therefore\ 6x+24-3x^2+6x+20-2x^2=160-72x+8x^2.$

$\therefore\ 13x^2-84x+116=0.$

$\therefore\ (13x-58)(x-2)=0.$

$\therefore\ x=2$ or $4\dfrac{6}{13}.$

6. Let x = the number of oranges bought; then the price of 1 orange $=\dfrac{384}{x}$ pies.

If he buys 8 more for the same money, the price of orange $=\dfrac{384}{x+8}$ pies.

By the question, $\dfrac{384}{x}-\dfrac{384}{x+8}=4.$

$\therefore\ 96\left(\dfrac{1}{x}-\dfrac{1}{x+8}\right)=1.$

$\therefore\ 96\times 8=x(x+8).$

$\therefore\ x^2+8x-768=0. \quad \therefore\ (x+32)(x-24)=0.$

$\therefore\ x=24.$

1894.

1. (2) $2(z^3+x^3)-[(x+y)(xy-x^2-y^2)-\{2(x+y+z)\times$
$(yz+zx+xy-x^2-y^2-z^2)-(x-y)(x^2+xy+y^2)\}]$

$=2(z^3+x^3)-[-x^3-y^3-\{-2(x^3+y^3+z^3-3xyz)$
$-(x^3-y^3)\}]$

$=2(z^3+x^3)+x^3+y^3-2(x^3+y^3+z^3-3xyz)-(x^3-y^3)$

$=6xyz.$ See Arts. 48 and 50.

2. $abx^2+(a^2+b^2)x+ab=(ax+b)(bx+a)$

and $(x^3+1)\div(x+1)=(x^2-x+1)$.

$\therefore$ quotient is $(bx+a)(x^2-x+1)$.

3. $x^4(x-a)^2+4a^2x^4+a^4(2x+a)^2-2a^3x^2(x+a)$.

Multiply out and arrange according to descending powers of x and extract the square root in the ordinary way.

4. See Example 2 art. 88 worked out for the H. C. F.

5. Taking the first and the second

$$\frac{b-c}{d-a}\left\{\frac{a-b}{c-d}-\frac{c-d}{a-b}\right\}=\frac{b-c}{d-a}\left\{\frac{(a-b)^2-(c-d)^2}{(a-b)(c-d)}\right\}$$

Taking the third and the fourth,

$$\frac{d-a}{b-c}\left\{\frac{c-d}{a-b}-\frac{a-b}{c-d}\right\}=\frac{d-a}{b-c}\left\{\frac{(c-d)^2-(a-b)^2}{(a-b)(c-d)}\right\}$$

$\therefore$ The given expression $=\dfrac{(a-b)^2-(c-d)^2}{(a-b)(c-d)}\left\{\dfrac{b-c}{d-a}-\dfrac{d-a}{b-c}\right\}$

$$=\frac{\{(a-b)^2-(c-d)^2\}\{(b-c)^2-(d-a)^2\}}{(a-b)(c-d)(d-a)(b-c)}.$$

$$=\frac{(a-b+c-d)(a-b-c+d)(b-c+d-a)(b-c-d+a)}{(a-b)(b-c)(c-d)(d-a)}.$$

6. (1) $\dfrac{x^2+3x+2}{x+3}=\dfrac{x^2+9x+20}{x+9}$

$\therefore\quad x+\dfrac{2}{x+3}=x+\dfrac{20}{x+9}$

$\therefore\quad \dfrac{1}{x+3}=\dfrac{10}{x+9}.\quad\therefore\quad x+9=10x+30.$

$\therefore\ 9x=-21.\qquad \therefore\ x=-2\frac{1}{3}.$

(3) $\sqrt{x^2+3x+15}=9-2x$

Squaring, $x^2+3x+15=81+4x^2-36x$.

$\therefore\ 3x^2-39x+66=0$

$\therefore\ x^2-13x+22=0$

$\therefore\ (x-11)(x-2)=0$

$\therefore\ x=11$ or 2.

7 Let $\dfrac{x}{y}$ be the fraction ;

then by the question, $\dfrac{x+3}{y+3}=\frac{2}{3}$ and $\dfrac{x-5}{y-5}=\frac{1}{2}$.

Solving these equations, we get $x=13$ and $y=21$.

8 Let x be the number of travellers and y the *no* of rooms ; then $x-y=a$ and $\dfrac{x}{2}=y-b$.

$\therefore\ a+y=2y-2b$. $\qquad\qquad \therefore\ y=a+2b$.

$\therefore\ x=2(a+b)$.

If they sleep 3 in a room, $\dfrac{2(a+b)}{3}$ rooms will be occupied

Therefore $a+2b-\dfrac{2(a+b)}{3}$ or $\dfrac{a+4b}{3}$ rooms will be empty.

1895.

1. $(4x^2-8x-1)^2-(2x^2-5x+7)^2$

$=(6x^2-13x+6)(2x^2-3x-8)\ \because\ a^2-b^2=(a+b)(a-b)$

$=(3x-2)(2x-3)(2x^2-3x-8)$.

$\therefore$ Quotient is $(3x-2)(2x-3)$.

2. (1) $\left(1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right)\left(1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right)$

$$=\left(\frac{x-5}{x-1}+\frac{12}{x-3}\right)\left(\frac{x+5}{x+1}-\frac{12}{x+3}\right)$$

$$=\frac{(x^2+4x+3)}{(x-1)(x-3)}\cdot\frac{(x^2-4x+3)}{(x+1)(x+3)}$$

$$=\frac{(x+1)(x+3)}{(x-1)(x-3)}\cdot\frac{(x-1)(x-3)}{(x+1)(x+3)}=1.$$

(2) $\dfrac{a(x-b)(x-c)}{bc(a-b)(a-c)}+\dfrac{b(x-c)(x-a)}{ac(b-c)(b-a)}+\dfrac{c(x-a)(x-b)}{ab(c-a)(c-b)}$

$$=\frac{1}{abc}\left\{\frac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\text{ the two similar terms.}\right\}$$

$$=\frac{1}{abc}\left[x^2\left\{\frac{a^2}{(a-b)(a-c)}+\right\}-x\left\{\frac{a^2(b+c)}{(a-b)(a-c)}+\dots\right\}\right.$$

$$\left.+abc\left\{\frac{a}{(a-b)(a-c)}+\dots\dots\right\}\right].$$

$$=\frac{1}{abc}\left[x^2\times 1-x\times 0+abc\times 0\right] \quad \text{(See Art. 102)}$$

$$=\frac{x^2}{abc}.$$

3. $3x^4-34x^2+51x-20$

$$=3x^3(x-1)+3x^2(x-1)-31x(x-1)+20(x-1).$$

(See Art. 70)

$$=(x-1)(3x^3+3x^2-31x+20).$$

Since $x-1$ is *not* a factor of the other expression, the H. C. F. of the given expressions is the same as that of $3x^3+3x^2-31x+20$ and $x^3+2x^2-3x+20$; which can be found by the ordinary method.

4. $x^6+2x^4+4x^3+x^2+4x-\frac{4}{x^2}-\frac{8}{x^3}+\frac{4}{x^6}$

$$=(x^3+x)^2+4(x^3+x)+4-\left(4+\frac{4}{x^2}+\frac{8}{x^3}\right)+\frac{4}{x^6}$$

$$=(x^3+x+2)^2-\frac{4}{x^3}(x^3+x+2)+\frac{4}{x^6}$$

$$=\left(x^3+x+2-\frac{2}{x^3}\right)^2 \quad \therefore \text{ Square root is } x^3+x+2-\frac{2}{x^3}.$$

N.B.—Students are recommended to use the ordinary method.

5. (1) $\frac{1}{x+1}+\frac{2}{2x+3}=\frac{6}{3x+5}$

$$\therefore\ \frac{1}{x+1}-\frac{2}{3x+5}=\frac{4}{3x+5}-\frac{2}{2x+3}$$

$$\therefore\ \frac{x+3}{(x+1)(3x+5)}=2\left\{\frac{x+1}{(3x+5)(2x+3)}\right\}$$

$$\therefore \quad \frac{x+3}{x+1} = \frac{2(x+1)}{2x+3}$$

$$\therefore \quad 2x^2 + 9x + 9 = 2x^2 + 4x + 2$$

$$\therefore \quad 5x = -7 \qquad \therefore \quad x = -1\tfrac{2}{5}.$$

(2) $$\frac{x-y}{a} + \frac{x+y}{b} = c \ldots\ldots\ldots\ldots(1).$$

and $$\frac{x-y}{b} - \frac{x+y}{a} = c \ldots\ldots\ldots\ldots(2).$$

From (1), $$(x-y)\frac{b}{a} + (x+y) = bc$$

From (2), $$(x-y)\frac{a}{b} - (x+y) = ac$$

$$\therefore (x-y)\left(\frac{b}{a} + \frac{a}{b}\right) = bc + ac$$

$$\therefore \quad x - y = \frac{abc(a+b)}{a^2+b^2} \ldots\ldots\ldots\ldots(3).$$

Again from (1), $$(x-y) + \frac{a}{b}(x+y) = ac$$

from (2), $$(x-y) - \frac{b}{a}(x+y) = bc$$

$$\therefore \quad (x+y)\left(\frac{a}{b} + \frac{b}{a}\right) = c(a-b)$$

$$\therefore \quad x+y = \frac{abc(a-b)}{a^2+b^2} \ldots\ldots\ldots\ldots(4).$$

$\therefore$ from (3) & (4), $$x = \frac{abc}{2}\left(\frac{2a}{a^2+b^2}\right) = \frac{a^2bc}{a^2+b^2}$$

and $$y = \frac{abc}{2}\left(\frac{-2b}{a^2+b^2}\right) = \frac{-ab^2c}{a^2+b^2}.$$

(3) $$141x^2 - 88x - 45 = 0.$$

$$\therefore \quad x^2 - \frac{88}{141}x - \frac{45}{141} = 0.$$

$$\therefore \quad x^2 - \frac{88}{141}x + \left(\frac{44}{141}\right)^2 = \left(\frac{44}{141}\right)^2 + \frac{45}{141}.$$

$$\therefore \quad \frac{19}{z} - \frac{2}{x} = 36$$

From (B), $\quad \dfrac{57}{x} + \dfrac{19}{z} = 95$

$$\therefore \quad \frac{59}{x} = 59. \qquad \therefore x = 1.$$

From (B), $\quad \dfrac{1}{z} = 2 \qquad \therefore z = \frac{1}{2}$

From (C), $\quad \dfrac{2}{y} = 16 - 10 = 6 \qquad \therefore y = \frac{1}{3}.$

(3) $\quad 129x^2 - 34x - 80 = 0.$

$$\therefore \quad x^2 - \frac{34}{129}x - \frac{80}{129} = 0.$$

$$\therefore \quad x^2 - \frac{34}{129}x + \left(\frac{17}{129}\right)^2 = \left(\frac{17}{129}\right)^2 + \frac{80}{129}$$

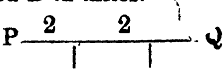
$$\therefore \quad \left(x - \frac{17}{129}\right)^2 = \left(\frac{17}{129}\right)^2 + \frac{80}{129} = \left(\frac{103}{129}\right)^2$$

$$\therefore \quad x - \frac{17}{129} = \pm \frac{103}{129}$$

$$\therefore \quad x = \frac{120}{129} \text{ or } -\frac{86}{129} = \frac{40}{43} \text{ or} -\frac{2}{3}.$$

6. Let x = the distance between A and B in miles.

And y = A's rate in miles per hour



z = B's..

They first meet at a distance of 2 miles from P.

$$\therefore \quad \frac{2}{y} = \frac{x-2}{z} \quad (1).$$

Again they meet at a distance of 4 miles from Q.

$$\therefore \quad \frac{6}{z} = \frac{x-2+x-4}{y} \quad i.e., \frac{3}{z} = \frac{x-3}{y} \ldots\ldots\ldots\ldots (2)$$

$\therefore$ Dividing (1) by (2), $\dfrac{x-2}{3} = \dfrac{2}{x-3}$.

$$\therefore \quad x^2 - 5x + 6 = 6. \qquad \therefore x = 5 \text{ miles.}$$

1897.

2. (2) $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3$

$= x^3(b-c+c-a+a-b)$

$-3x^2\{a(b-c)+b(c-a)+c(a-b)\}$

$+3x\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$

$-\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}$

$= x^3\times 0-3x^2\times 0+3x(a-b)(b-c)(a-c)$

$\quad -(a-b)(b-c)(a-c)(a+b+c)$

(See Arts 74 and 76).

$=(a-b)(b-c)(c-a)(a+b+c-3x)$

Divisor is $(a-b)(b-c)(c-a)$

$\therefore$ Quotient $=a+b+c-3x$

3 (1) See example 3, Art. 63

(2) See example 5, do.

(3) Multiplying, we have,

$x^2-3x+2-2(xy-x-2y+2)+y^2-3y+2$

$= x^2+y^2-2xy-x+y=(x-y)^2-(x-y)$

$=(x-y)(x-y-1).$

4 (1) $\dfrac{(b+c-2a)^3-(c+a-2b)^3}{(c+a-2b)^3-(a+b-2c)^3}=\dfrac{y^3-z^3}{z^3-x^3}$, putting x for $a+b-2c$, y for $b+c-2a$, z for $c+a-2b$.

We see that $x+y+z=a+b-2c+b+c-2a+c+a-2b=0$

$\therefore$ $y^2+yz+z^2=z^2+zx+x^2$..... (See example II (ii) of Art 79)

$\therefore$ The given expression $=\dfrac{(y-z)(y^2+yz+z^2)}{(z-x)(z^2+zx+x^2)}$

$=\dfrac{y-z}{z-x}=\dfrac{(b+c-2a)-(c+a-2b)}{(c+a-2b)-(a+b-2c)}=\dfrac{3(b-a)}{3(c-b)}=\dfrac{a-b}{b-c}.$

(2) $\dfrac{a+b}{a-b+\dfrac{b^2}{a+b}}-\dfrac{a-b}{a+b+\dfrac{b^2}{a-b}}$

$$=\frac{a+b}{\frac{a^2}{a+b}}-\frac{a-b}{\frac{a^2}{a-b}}=\frac{(a+b)^2}{a^2}-\frac{(a-b)^2}{a^2}$$

$$=\frac{4ab}{a^2}=\frac{4b}{a}.$$

5 $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0 \quad \therefore xy+yz+zx=0$

Now $\frac{1}{2}\left(\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}+\frac{x^2y^2}{z^2}\right)=\frac{1}{2}\left(\frac{y^4z^4+z^4x^4+x^4y^4}{x^2y^2z^2}\right)$

$=\frac{1}{2}\left\{\frac{2(yz\times zx+zx\times xy+xy\times yz)^2}{x^2y^2z^2}\right\}$ See example I (iii) Art. 79.

[If $a+b+c=0$, $a^4+b^4+c^4=2(ab+ac+bc)^2$]

$$=\left(\frac{xyz^2+yzx^2+xzy^2}{xyz}\right)^2=(z+x+y)^2.$$

6. (1) $\frac{1}{3}(x-4)+\frac{3}{7}(2x-7)-\frac{7}{9}(1+5x)=4(1-x)$

$\therefore \frac{x}{3}+\frac{6x}{7}-\frac{35x}{9}+4x=\frac{4}{3}+3+\frac{7}{9}+4$

$x(\frac{1}{3}+\frac{6}{7}-\frac{35}{9}+4)=\frac{82}{9}$

$x(\frac{82}{63})=\frac{82}{9} \qquad x=7$

(2) $\frac{4}{2x+3}-\frac{18}{7x+12}=5$

$\therefore 28x+48-36x-54=5(14x^2+45x+36)$

$70x^2+233x+186=0$

$\therefore x=\frac{233\pm\sqrt{233^2-4\times70\times186}}{140}$

Again the $\therefore x=\frac{233\pm47}{140}=-2$ or $-1\frac{23}{70}$

$\therefore \frac{6}{z}=$

$3x-4y+5z=-26\ldots$(A).

$\therefore$ Dividi $3y-4z+5x=13\ldots$(B).

$-4x+5y=5\ldots$(C).

$\therefore x^2-5x+$

Adding A, B and C, we have,

$$4x+4y+4z=-8$$

$$\therefore\ x+y+z\quad=-2\ldots(\mathrm{D}).$$

From (A), $3x-4y+5z=-26$.

From 4 (D), $4x+4y+4z=-8$

$$\therefore\quad 7x+9z=-34\ldots(\mathrm{E}).$$

Again, $3y-4z+5x=13$

& 3 (D) is $3x+3y+3z=-6$

$$\therefore\quad 2x-7z=19\ldots(\mathrm{F}).$$

From E, $14x+18z=-68$

From F, $14x-49z=133$.

$$\therefore\quad 67z=-201\ldots z=-3$$

and $x=\frac{1}{2}(19-21)=-1$ (from F)

and $y=-2+3+1=2$ (from D).

7. Let $x=$ the price of 1lb. of tea in *annas*

and $y=$ .... sugar ....

then the price of 3 lbs. of tea and 4 lbs. of sugar

$=(3x+4y)$ annas.

By the question, $3x+4y=56\ldots\ldots(\mathrm{A})$

Again, the *increase* in the price of 3 lbs of tea and 4 lbs. of sugar $=(3\times\frac{11}{10}x+4\times\frac{3}{2}y)-(3x+4y)$ as. $=(\frac{3}{10}x+y)$ annas.

By the question, $\frac{3x}{10}+2y=12\ldots(\mathrm{B})$.

From A, $3x+4y=56$.

From B, $\frac{3x}{5}+4y=24$.

$$\therefore\quad \frac{12}{5}x=32\quad\therefore\ x=\frac{5}{12}\times 32=\frac{40}{3}\text{ as.}=13\text{ as. }4\text{ p.}$$

1898.

1. (a) $(x^2-1)^4-3(x^2-1)^2+1$

$=y^4-3y^2+1$ putting $y=x^2-1$

$=(y^2-1)^2-y^2=(y^2+y-1)(y^2-y-1)$

$\{(x^2-1)^2+(x^2-1)-1\}\{(x^2-1)^2-(x^2-1)-1\}$

$=(x^4-x^2-1)(x^4-3x^2+1)$

$\therefore$ the quotient is x^4-x^2-1.

2. (1) $(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2$

$=4a^2b^2-(a^2+b^2-c^2)^2$

$=(a+b+c)(a+b-c)(c+a-b)(c-a+b)$ *Vide* Art 63, (Ex. 5).

(2) $a^3(a+1)+b^3(b+1)-ab(a-b)^2$

$=a^4+b^4+a^3+b^3-ab(a^2+b^2)+2a^2b^2$

$=(a^2+b^2)^2-ab(a^2+b^2)+a^3+b^3$

$=(a^2+b^2)(a^2+b^2-ab)+(a+b)(a^2-ab+b^2)$

$=(a^2+b^2-ab)(a^2+b^2+a+b)$.

3. By the ordinary method, the H. C. F. can be found

4. (a) $\dfrac{1}{x^3-x^2+x-1}=\dfrac{2}{(x-1)(x^2+1)}$; $\dfrac{3}{2x^2-x-1}$

$=\dfrac{3}{(2x+1)(x-1)}$; and $\dfrac{1-3x}{2x^3+x^2+2x+1}=\dfrac{1-3x}{(2x+1)(x^2+1)}$

The given Expn. $=\dfrac{2x+1+3(x^2+1)+(1-3x)(x-1)}{(x-1)(2x+1)(x^2+1)}$

$=\dfrac{6x+3}{(x-1)(2x+1)(x^2+1)}=\dfrac{3}{(x-1)(x^2+1)}$.

(b) The numerator —

$=(3a+b)^3\{(3a+b)-(2a+6b)\}+(a+3b)^3\{6a+2b-(a+3b)\}$

$=(3a+b)^4-2(a+3b)(2a+b)^3+2(a+3b)^3(3a+b)-(a+3b)^4$

$=(3a+b)^4-(a+3b)^4-2(a+3b)(3a+b)\{(3a+b)^2-(a+3b)^2\}$

$=\{(3a+b)^2-(a+3b)^2\}\{(3a+b)^2+(a+3b)^2-2(a+3b)(3a+b)\}$

$=4(a+b)2(a-b)\{3a+b-(a+3b)\}^2$

$=8(a+b)(a-b)4(a-b)^2=32(a+b)(a-b)^3$

$\therefore$ the given expression $=32(a-b)^3$.

5. Arrange the expression according to descending powers of a or b and extract the square root by the ordinary method.

6. (1) $1+\frac{1}{2x-8}-\left(1-\frac{1}{3x+9}\right)=\frac{x-3}{(x-4)(x+3)}$

$\therefore \frac{1}{2(x-4)}+\frac{1}{3(x+3)}=\frac{x-3}{(x-4)(x+3)}$

$\therefore 3(x+3)+2(x-4)=6(x-3)$

$\therefore 5x+1=6x-18 \qquad \therefore x=19.$

(2) $x+\frac{ay}{a-b}=b \quad \ldots.(1)$

$y+\frac{ax}{a+b}=b \quad \ldots..(2).$

From (1), $x+y\cdot\frac{a}{a-b}=b$

From (2), $y\cdot\frac{a}{a-b}+x\frac{a^2}{a^2-b^2}=\frac{ab}{a-b}$,

$\therefore x\left\{\frac{a^2}{a^2-b^2}-1\right\}=\frac{ab}{a-b}-b$

$\therefore x\frac{b^2}{a^2-b^2}=\frac{b^2}{a-b}$

$\therefore x=a+b$, and $y=b-\frac{ax}{a+b}=b-a$

(3) $1-\frac{1}{x-2}+1-\frac{5}{x+1}+\frac{1}{4}=0$

$\therefore \frac{9}{4}=\frac{1}{x-2}+\frac{5}{x+1}=\frac{6x-9}{(x-2)(x+1)}$

$\therefore 3(x-2)(x+1)=4(2x-3)$

$\therefore 3x^2-3x-6=8x-12$

$\therefore 3x^2-11x+6=0 \qquad \therefore x=\frac{11\pm\sqrt{121-72}}{6}$

$=\frac{11\pm\sqrt{39}}{6}.$

7. Let x= the length in *yards* and y= the breadth in *yards*; then xy= the area of the room.

By the question, $(x+3)(y-3)=xy-18\ldots\ldots(1)$
and $(x+3)(y+3)=xy+60\ldots\ldots(2)$.

From (1), $3(y-x)=-9$; and from (2), $3(x+y)=51$
$\therefore x+y=17$ and $x-y=3$ $\therefore x=10$ yds. and $y=7$ yds.

1899.

2. (1) $(ab+1)^4-4ab(ab+1)^2-(a^2-b^2)^2$
$=(ab+1)^4-4ab(ab+1)^2+4a^2b^2-(a^2-b^2)^2-4a^2b^2$
$=\{(ab+1)^2-2ab\}^2-(a^2+b^2)^2$
$=(a^2b^2+1)^2-(a^2+b^2)^2$
$=(a^2b^2+1+a^2+b^2)(a^2b^2+1-a^2-b^2)$
$=(a^2+1)(b^2+1)(a^2-1)(b^2-1)$
$=(a+1)(a-1)(b+1)(b-1)(a^2+1)(b^2+1)$

(2) $(a^2-b^2)(a+b)+(b^2-c^2)(b+c)+(c^2-a^2)(c+a)$
$=(a+b+c)\{a^2-b^2+b^2-c^2+c^2-a^2\}$
$-c(a^2-b^2)-a(b^2-c^2)-b(c^2-a^2)$
$=-\{c(a^2-b^2)+a(b^2-c^2)+b(c^2-a^2)\}$
$=-(a-b)(b-c)(c-a)=(a-b)(b-c)(a-c)$. *Vide* Art. 74, Ex. 1.

3. Find the G. C. M. of the numerator and the denominator and divide them by the G. C. M.

4. The numerator $=\{a+x+2(x+b)\}(a+x)^3$
$-\{2(a+x)+b+x\}(b+x)^3$
$=(a+x)^4+2(a+x)^3(b+x)-2(a+x)(b+x)^3-(b+x)^4$
$=(a+x)^4-(b+x)^4+2(a+x)(b+x)\{(a+x)^2-(b+x)^2\}$
$=\{(a+x)^2-(b+x)^2\}\{(a+x)^2+(b+x)^2+2(a+x)(b+x)\}$

$=($ [illegible] $)\{a+x+b+x\}^2$
$=(a$ [illegible]
$\therefore$ the g[illegible] $=a-b$.

5. (a) $x^6+\frac{1}{x^6}$ [illegible] $\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20.$

$$=x^6+\frac{1}{x^6}+2+6\left(x^4+\frac{1}{x^4}+x^2+\frac{1}{x^2}\right)$$
$$+9\left(x^2+\frac{1}{x^2}+2\right)$$
$$=\left(x^3+\frac{1}{x^3}\right)^2+6\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)+9\left(x+\frac{1}{x}\right)^2$$
$$=\left\{x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)\right\}^2$$

$\therefore$ the square root is $x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$

(b) We know that

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a).$$

$$\therefore \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3=\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}+3\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{z}+\frac{1}{x}\right)$$

$$\therefore \frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3-\frac{3(x+y)(y+z)(z+x)}{x^2y^2z^2}$$

but $3(x+y)(y+z)(z+x)=3(-z)(-x)(-y)=-3xyz$ because $x+y+z=0$

$$\therefore \frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3+\frac{3}{xyz}.$$

6. (1) $\frac{x+5}{2x-1}-\frac{3(5x+1)}{5x+4}=\frac{4}{2x-1}-2\frac{1}{2}$

$\therefore \quad \frac{x+5-4}{2x-1}=\frac{3(5x+1)}{5x+4}-2\frac{1}{2}$

$\therefore \quad \frac{x+1}{2x-1}=\frac{6(5x+1)-5(5x+4)}{2(5x+4)}$

$\therefore \quad 2(x+1)(5x+4)=(2x-1)(5x-14)$

$\therefore \quad 10x^2+18x+8=10x^2-33x+14$

$\therefore \quad 51x=6 \quad \therefore \quad x=\frac{2}{17}$

(2) $1+\frac{2}{x-1}+1+\frac{4}{x-2}=2+\frac{11}{x+1}$

$$\therefore \quad \frac{2}{x-1}+\frac{4}{x-2}=\frac{11}{x+1}$$

$$\therefore \quad 2(x-2)(x+1)+4(x-1)(x+1)=11(x-1)(x-2)$$

$$\therefore \quad 2x^2-2x-4+4x^2-4=11x^2-33x+22$$

$$\therefore \quad 5x^2-31x+30=0$$

$$\therefore \quad (5x-6)(x-5)=0$$

$$\therefore \quad x=5 \text{ or } 1\tfrac{1}{5}.$$

(3) $3x-4y-6z=-16$............(A)

$4x-y-z=5$.................(B)

$x-3y-4z=-12$.................(C)

From (A), $3x-4y-6z=-16$

From (B), $16x-4y-4z=20$

$\therefore$ $13x+2z=36$ ....(D)

Again, from (C), $x-3y-4z=-12$

from (B), $12x-3y-3z=15$

$\therefore$ $11x+z=27$. . ...(E)

$\therefore$ $22x+2z=54$

from (D), $13x+2z=36$

$\therefore$ $9x=18$ $\therefore$ $x=2$

and $z=27-11x=5$ (from E)

and $y=4x-z-5=8-5-5=-2$ (from B)

7 Let $x=$ the distance in miles

and $y=$ A's rate in miles per hour,

then $y-\frac{1}{2}=$ B's rate and $y-\frac{1}{4}=$ C's rate.

By the question, $\dfrac{x}{y-\frac{1}{2}}=\dfrac{x}{y-\frac{1}{4}}+\frac{3}{4}$... ....(1)

and $\dfrac{x}{y-\frac{1}{4}}=\dfrac{x}{y}+2$.. .... (2)

From (1), $-\frac{1}{4}x+\frac{1}{2}x=\frac{3}{4}(y-\frac{1}{2})(y-\frac{1}{4})$

$\therefore$ $x=3(y-\frac{1}{2})(y-\frac{1}{4})$... ....(3)

From (2), $\frac{1}{4}x=2y(y-\frac{1}{4})$

$\therefore$ $x=8y(y-\frac{1}{4})$.................................(4)

$\therefore$ from (3) and (4), $3(y-\frac{1}{2})(y-\frac{1}{4})=8y(y-\frac{1}{4})$

$\therefore\ 9(y-\frac{1}{2})=8y.$

$\therefore\ y=4\frac{1}{2}$ miles

Hence A's rate $=4\frac{1}{2}$ miles, B's rate $=4$ miles and C's rate $=3\frac{3}{4}$ miles.

N.B.—If $y-\frac{1}{4}=0$, then C's rate is Zero. Hence that value was not considered.

March 1900

2 (1) $(b+c)^2-2(b^2+c^2)a^2+(b-c)^2a^4$

$=(b+c)^2-a^2\{(b+c)^2+(b-c)^2\}+(b-c)^2a^4$

$=(b+c)^2(1-a^2)-(b-c)^2a^2(1-a^2)$

$=(1-a^2)\{(b+c)^2-a^2(b-c)^2\}$

$=(1+a)(1-a)(b+c+ab-ac)(b+c-ab+ac).$

(2) $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

$=a^3(b^2-c^2)+b^3(c^2-b^2+b^2-a^2)+c^3(a^2-b^2)$

$=a^3(b^2-c^2)-b^3(b^2-c^2)-b^3(a^2-b^2)+c^3(a^2-b^2)$

$=(b^2-c^2)(a^3-b^3)-(a^2-b^2)(b^3-c^3)$

$=(a-b)(b-c)\{(b+c)(a^2+ab+b^2)$
$\qquad -(a+b)(b^2+bc+c^2)\}$

$=(a-b)(b-c)\{(b+c)a^2+b(a+b)(b+c)-c^2(a+b)$
$\qquad -b(b+c)(a+b)\}$

$=(a-b)(b-c)\{(b+c)a^2-(a+b)c^2\}$

$=(a-b)(b-c)\{b(a^2-c^2)+ac(a-c)\}$

$=(a-b)(b-c)(a-c)(ab+bc+ac).$

3 The H. C. F. is easily found by the ordinary method.

4 (1) $\dfrac{1}{x^4+2x^3}+\dfrac{1}{x^4-2x^3}+\dfrac{2}{x^4+4x^2}$

$=\dfrac{1}{[illegible]}+\dfrac{1}{x^3(x-2)}+\dfrac{2}{x^2(x^2+4)}$

$=\dfrac{[illegible]}{x^2\ [illegible]}\ \dfrac{2}{[illegible](x^2+4)}$

$=\dfrac{[illegible]}{x^2(x^2-[illegible]}\ [illegible](x^2+4)}$

$$= \frac{2(x^2+4)+2(x^2-4)}{x^2(x^4-16)} = \frac{4}{x^4-16}.$$

(2) Numerator $= \frac{(x-a)^3+(x+a)^3}{(x^2-a^2)^3} = \frac{2(x^3+3xa^2)}{(x^2-a^2)^3}.$

Denominator $= \frac{1}{(a+x)^2} + \frac{1}{a^2-x^2} + \frac{1}{(a-x)^2}$

$$= \frac{(a-x)^2+a^2-x^2+(a+x)^2}{(a^2-x^2)^2}$$

$$= \frac{3a^2+x^2}{(a^2-x^2)^2}$$

Hence the given Expression $= \frac{2(x^3+3xa^2)}{(x^2-a^2)^3} \times \frac{(a^2-x^2)^2}{3a^2+x^2}$

$= 2x.$

5. $(a^2+3b^2)^2+10ab(a+b)(a-3b)+33a^2b^2.$

$=(a^2+3b^2)^2-12a^2b^2+10ab\{(a^2-3b^2-2ab)+33a^2b^2+12a^2b^2$

$=(a^2-3b^2)^2+10ab\,(a^2-3b^2)-20a^2\,b^2+45a^2\,b^2$

$=(a^2-3b^2)^2+2\times5ab(a^2-3b^2)+25a^2\,b^2$

$=(a^2-3b^2+5ab)^2.$ Hence Sq. root $=a^2+5ab-3b^2.$

6. (1) $\frac{2x-3}{5}+\frac{9}{10}(3x+8)=5x+\frac{1}{3}(4x-19)$

Multiply both sides by 30.

$\therefore\ 6\,(2x-3)+27(3x+8)=150x+10(4x-19)$

$\therefore\ 12x-18+81x+216=190x-190$

$\therefore\ 190x-93x=216+190-18$

$\therefore\ 97x=388\ \therefore\ x=4$

(2) $\frac{3x-1}{3x-9}+\frac{x+1}{x-7}=\frac{x-10}{x^2-10x+21}$

$\therefore\ \frac{(3x-1)(x-7)+3(x-3)(x+1)}{3(x^2-10x+21)}=\frac{x-10}{x^2-10x+21}$

$\therefore\ (3x-1)(x-7)+3(x-3)(x+1)=3(x-10)$

$\therefore\ 3x^2-22x+7+3x^2-6x-9=3x-30$

$\therefore\ 6x^2-31x+28=0$

$\therefore\ (6x-7)(x-4)=0$

$\therefore\ x=4$ or $1\frac{1}{6}.$

(3) $x+2y+3z=\frac{19}{6}$.........(A)

$2x+3y+z=2$..........(B)

$3x-4y-7z=\frac{1}{6}$........(C)

From (B), $2x+3y+z=2$

From (A), $2x+4y+6z=\frac{19}{3}$

$\therefore\ y+5z=\frac{13}{3}$.......................(D)

Again, from (C), $3x-4y-7z=\frac{1}{6}$

and from (A), $3x+6y+9z=\frac{19}{2}$

$\therefore\ 10y+16z=\frac{56}{6}=\frac{28}{3}$

$\therefore\ 5y+8z=\frac{14}{3}$............(E)

From (D), $5y+25z=\frac{65}{3}$

$\therefore\ 17z=\frac{51}{3}=17 \quad \therefore\ z=1$

and $y=\frac{13}{3}-5z=-\frac{2}{3}$ (from D)

and $x=\frac{19}{6}-2y-3z=\frac{19}{6}+\frac{4}{3}-3=1\frac{1}{2}$ (from A).

7. Let $x=$ the amount in rupees he has
and $y=$ the number of people;
then, by the conditions of the question,

$2y=x-20$.............. (1)

and $\frac{x}{2\frac{1}{2}}=y-10$ ....(2)

$\therefore\ x=2\frac{1}{2}y-25$

and from (1), $x=2y+20$

$\therefore\ 0=\frac{1}{2}y-45 \qquad \therefore\ y=90$ people

and $x=2y+20=180+20=200$

He has Rs. 200 to give away among 90 people.

DECEMBER 1900.

1. (1) $\frac{1+x^3}{(1+x)^2}+\frac{1-x^3}{(1-x)^2}$

$=\frac{1-x+x^2}{(1+x)}+\frac{1+x+x^2}{(1-x)}$

$=\frac{(1-x)^2+x^2(1-x)+(1+x)^2+x^2(1+x)}{1-x^2}$

$2x-5y+z=1\ldots \quad \text{(C)}$

Subtracting (C) from (B), $4y-3z=1\ldots$(D).

Again, from (A), $2x+4y-6z=2$

and from (B), $2x-\ y-2z=2$

$\therefore\ 5y-4z=0\ldots$(E).

From (D), $20y-15z=5$

and from (E), $20y-16z=0$

$\therefore\ z=5$: and $y=\dfrac{4z}{5}=4$ (from E).

and $x=1-2y+3z=1-8+15=8$ (from A)

7. Let x = the number of lbs. of the best quality

then $100-(28+x)$ or $72-x$ = the number of lbs. of the intermediate quality.

The price of x lbs. of the best quality $=\frac{3}{2}x$ (rupees)

... $72-x$ lbs...2nd... $=(72-x)\frac{9}{8}$ (rupees)

and...28 lbs...3rd... $=28\times\frac{7}{8}=$ Rs. $\frac{49}{2}$.

$\therefore$ the price of 100 lbs. of the mixture $=\frac{3}{2}x+\frac{9}{8}(72-x)+\frac{49}{2}$

By the question, the price of the same $=100\times$ Rs. $\frac{5}{4}$

$=$ Rs. 125

$\therefore\ \frac{3}{2}x+\frac{9}{8}(72-x)+\frac{49}{2}=125$

$\therefore\ \frac{3}{2}x-\frac{9}{8}x=125-81-\frac{49}{2}=19\frac{1}{2}$.

$\therefore\ \frac{3}{8}x=\frac{39}{2}$

$\therefore\ x=52$lbs.

December 1901.

1. (1) Find the coefficient of x^5 in $(1+3x-4x^2)^4$ and of x^8 in $(x+x^3+x^5+x^6)^3$.

(2) Divide $a^2x^6-3(3b^2-2ac)x^4+3(3c^2-2bd)x^2-d^2$ by $ax^3+3bx^2+3cx+d$.

2. Resolve into elementary factors :—

(1) $x^4-65x^2y^2+64y^4$.

(2) $x^5+y^5-x^4y-xy^4$.

(3) $(5a+3b)^3-(3a+5b)^3-8(a-b)^3$.

3. Find the L. C. M. of the expressions $3x^3-11x^2+6x$ and $x^5-9x^4+27x^3-27x^2$.

4. Simplify

(1) $x(y-z)(y+z-x)^2+y(z-x)(z+x-y)^2$
$$+z(x-y)(x+y-z)^2,$$

(2) $$\frac{3x-\frac{8}{x}}{x^2-\frac{1}{x}}-\frac{5+\frac{7}{x}}{x+1+\frac{1}{x}}+\frac{2x^2+2x^4}{x^4-x^2}.$$

5. Prove the following identities —

(1) $a^2(a^4-10a^2x^2+5x^4)^2+x^2(5a^4-10a^2x^2+x^4)^2$
$$=(a^2+x^2)^5.$$

(2) $$\frac{(y-z)^5+(z-x)^5+(x-y)^5}{(y-z)^3+(z-x)^3+(x-y)^3}$$
$$=\tfrac{5}{3}\{x^2-x(y+z)+y^2-yz+z^2\}$$

6. Solve the following equations :—

(1) $$\frac{x-2a}{a+4b}+\frac{x-2b}{3a+2b}+\frac{x}{3a+4b}=3$$

(2) $$\frac{2}{x+1}+\frac{3}{x+3}=\frac{2}{2x+1}+\frac{4}{x+4}$$

(3) $x-4y=\frac{1}{5}(4x-7y)-3=\frac{2}{3}(2x-6y+1)$

7. In a rectangular picture frame, 3 feet by 4, one-eighth of the whole area is occupied by the frame, which is of uniform width all round, and the remainder by the glass, show that the width of the frame is 1·33 inches nearly.

1. (2) $a^2x^6-3(3b^2-2ac)x^4+3(3c^2-2bd)x^2-d^2$

$=(a^2x^6+6acx^4+9c^2x^2)-(9b^2x^4+6bdx^2+d^2)$

$=(ax^3+3cx)^2-(3bx^2+d)^2$

$=(ax^3+3cx+3bx^2+d)(ax^3+3cx-3bx^2-d)$

∴ the quotient is $ax^3-3bx^2+3cx-d$

2. (1) $x^4-65x^2y^2+64y^4=(x^2-y^2)(x^2-64y^2)$

Vide. Art. 70.

$=(x+y)(x-y)(x+8y)(x-8y)$.

(2) $x^5+y^5-x^4y-xy^4=x^4(x-y)-y^4(x-y)$
$=(x-y)(x^4-y^4)=(x-y)^2(x+y)(x^2+y^2)$.

(3) $(5a+3b)^3-(3a+5b)^3-8(a-b)^3$.
$=(5a+3b)^3+(-3a-5b)^3+(-2a+2b)^3$
$=3(5a+3b)(-3a-5b)(-2a+2b)$ See Art 77.
$=6(5a+3b)(3a+5b)(a-b)$

N.B. If $p+q+r=0$, then $p^3+q^3+r^3=3pqr$.

3. $3x^3-11x^2+6x=x(3x^2-11x+6)=x(3x-2)(x-3)$.

And $x^5-9x^4+27x^3-27x^2=x^2(x^3-9x^2+27x-27)$
$=x^2(x-3)^3$

∴ L. C. M. is $x^2(x-3)^3(3x-2)$.

4. (1) Let $y+z-x=a$; $z+x-y=b$; and $x+y-z=c$, then $a+b=2z$, $b+c=2x$; $c+a=2y$.

And $a-b=2(y-x)$; $b-c=2(z-y)$; $c-a=2(x-z)$

∴ the given expression becomes

$$\frac{b+c}{2}\times\frac{-(b-c)}{2}a^2+\frac{c+a}{2}\times\frac{-(c-a)}{2}b^2+\frac{a+b}{2}\times\frac{-(a-b)}{2}c^2.$$

$$=-\tfrac{1}{4}\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\}=0$$

(2) Expression $=\dfrac{3x^2-8}{x^3-1}-\dfrac{5x+7}{x^2+x+1}+\dfrac{2(x^2+1)}{x^2-1}$

$$=\frac{(3x^2-8)(x+1)-(x^2-1)(5x+7)+2(x^2+1)(x^2+x+1)}{(x^3-1)(x+1)}$$

$$=\frac{3x^3+3x^2-8x-8-5x^3-7x^2+5x+7+2x^4+4x^2+2+2x^3+2x}{(x^3-1)(x+1)}$$

$$=\frac{2x^4-x+1}{(x^3-1)(x+1)}.$$

5. (1) $(a^2+x^2)^5=(a+ix)^5(a-ix)^5$ where $i=\sqrt{-1}$

$=(a^5+5a^4ix+10a^3i^2x^2+10a^2i^3x^3+5ai^4x^4+i^5x^5)$

$\times(a^5-5a^4ix+10a^3i^2x^2-10a^2i^3x^3+5ai^4x^4-i^5x^5)$

$=\{(a^5-10a^3x^2+5ax^4)+i(x^5-10a^2x^3+5ax)\}$

$\{(a^5-10a^3x^2+5ax^4)-i(x^5-10a^2x^3+5a^4x)\}$.

Since $i^2=-1$; $i^3=-i$; $i^4=1$; $i^5=i$

$=(a^5-10a^3x^2+5ax^4)^2-i^2(5a^4x-10x^3a^2+x^5)^2$

$=a^2(a^4-10a^2x^2+5x^4)^2+x^2(5a^4-10a^2x^2+x^4)^2.$

(2) If $a+b+c=0$, then $a^3+b^3+c^3=3abc$

and $a^5+b^5+c^5=\frac{5}{6}(a^3+b^3+c^3)(a^2+b^2+c^2)$ See Art. 77 and Art. 165, Ex. 9.

Now in the question, $x-y+y-z+z-x=0$

i.e., $a+b+c=0$ if $x-y=a$, $y-z=b$ and $z-x=c$.

$\therefore$ L.H.S. of the Expression $=\dfrac{a^5+b^5+c^5}{a^3+b^3+c^3}$

$=\frac{5}{6}(a^2+b^2+c^2)$

$=\frac{5}{6}\{(x-y)^2+(y-z)^2+(z-x)^2\}$

$=\frac{5}{3}(x^2+y^2+z^2-xy-yz-zx)$

$=\frac{5}{3}\{x^2-x(y+z)+y^2-yz+z^2\}.$

6. (1) $\dfrac{x-2a}{a+4b}+\dfrac{x-2b}{3a+2b}+\dfrac{x}{3a+4b}=3.$

$\therefore\ \dfrac{x-2a}{a+4b}-1+\dfrac{x-2b}{3a+2b}-1+\dfrac{x}{3a+4b}-1=0$

$\therefore\ \dfrac{x-3a-4b}{a+4b}+\dfrac{x-3a-4b}{3a+2b}+\dfrac{x-3a-4b}{3a+4b}=0.$

$\therefore\ x-3a-4b=0\quad\therefore\ x=3a+4b.$

See Art. 139, Ex. 8.

(2) $\dfrac{2}{x+1}+\dfrac{3}{x+3}=\dfrac{2}{2x+1}+\dfrac{4}{x+4}$

$\therefore\ \dfrac{2}{x+1}-\dfrac{2}{2x+1}=\dfrac{4}{x+4}-\dfrac{3}{x+3}$

$\therefore\ \dfrac{2x}{(x+1)(2x+1)}=\dfrac{x}{(x+4)(x+3)}$

$\therefore\ x=0$ *or*

$\dfrac{2}{(x+1)(2x+1)}=\dfrac{1}{(x+4)(x+3)}$

$\therefore\ 2x^2+14x+24=2x^2+3x+1$

$\therefore\ 11x=-23\ \therefore\ x=-2\frac{1}{11}.$

(3) $5(x-4y)=4x-7y-15$...............(A).

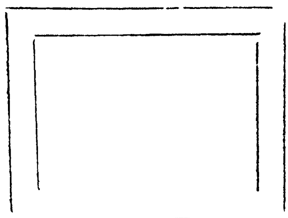
$3(x-4y)=2(2x-6y+1)$. ....(B).

$\therefore$ from (A), $x-13y+15=0$............(C).

from (B), $x+2 \qquad\qquad =0$........ ...

$\therefore$ $x=-2$ and from (C), $-13y=-13$ $\therefore$ $y=1$.

7. Let the width of the frame be x inches, then the length of the glass $=(48-2x)$ inches and the breadth $=(36-2x)$ inches.



By the question, $(48-2x)(36-2x)=\frac{7}{8}\times 48\times 36$

$\therefore$ $(x-24)(x-18)=7\times 6\times 9=378$

$\therefore$ $x^2-42x+432=378$

$\therefore$ $x^2-42x+54=0$

$$\therefore x=\frac{42\pm\sqrt{(42)^2-216}}{2}=21\pm\sqrt{441-54}$$

$$=21\pm\sqrt{387}=21\pm 19\cdot 67\ldots.$$

$$=40\cdot 67 \text{ or } 1\cdot 33\ldots\ldots$$

The former value is *obviously* inadmissible

$\therefore$ the width of the frame $=1\cdot 33$ inches.

(3) $5(x-4y)=4x-7y-15$ ....(A).
$3(x-4y)=2(2x-6y+1)$. ....(B).
$\therefore$ from (A), $x-13y+15=0$............(C).
from (B), $x+2 \qquad\qquad =0$........ ...
$\therefore$ $x=-2$ and from (C), $-13y=-13$ $\therefore$ $y=1$.

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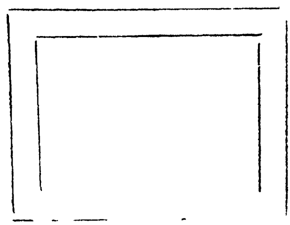

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$\therefore$ $x^2-42x+432=378$
$\therefore$ $x^2-42x+54=0$

$$\therefore x=\frac{42\pm\sqrt{(42)^2-216}}{2}=21\pm\sqrt{441-54}$$
$$=21\pm\sqrt{387}=21\pm 19{\cdot}67\ldots.$$
$$=40{\cdot}67 \text{ or } 1{\cdot}33\ldots\ldots$$

The former value is *obviously* inadmissible
$\therefore$ the width of the frame $=1{\cdot}33$ inches.

ANSWERS TO EXERCISES.

EXERCISE 1 (*A*).

1. 61. 2. 1. 3. 33. 4. 345. 5. -444.
6. 2082. 7. 1034. 8. 517. 9. 43. 10. -402.

EXERCISE 1 (*B*).

1. $6\frac{11}{12}$. 2. -1. 3. $-\frac{32}{15}$. 4. 29. 5. 192.
6. 58. 7. $17\frac{9}{10}$. 8. 10. 9. 11. 10. $3\frac{61}{630}$.

EXERCISE 1 (*C*).

1. 3. 2. 55. 3. 4. 4. $\frac{365}{741}$. 5. -108.
6 5. 7. 0. 8. 76 ; 536. 9. 38. 10. 50.

EXERCISE 2.

1. $8a$; $-6b$; $2a$.
2. $37ab$; $13a^2b$.
3. $17\sqrt{xy}$; $-5\frac{1}{2}\sqrt{ab}$.
4. $23a-29b$.
5. $6a^2-8ab+11b^2$.
6. $2\frac{1}{4}x^2-3\frac{1}{4}xy+5\frac{1}{2}y^2$.
7. $11\frac{1}{2}-15\frac{1}{3}ax+8\frac{1}{2}by-7ab$.

EXERCISE 3.

1. $3(a+b+c+d)$.
2. $-5a-a^3+6a^2+2$.
3. $2ax-4by+2cz$.
4. $2a^4+12a^2+2$.
5. $4a^3-\frac{1}{2}a^2x+\frac{7}{6}ax^2+\frac{1}{12}x^3$.
6. $15a^m+33b^n-29c^p+37d$.

EXERCISE 4.

1. $a+b+c-x-y$.
2. $y^2(a+b+c)+cy+c$.
3. $2(x^2+y^2+z^2)$.
4. $6a^3-2a^2b-3ab^2-22ac^2-c^3-7b^3$.
5. $8\frac{1}{2}a-5\frac{47}{60}b+3\frac{1}{8}c-3\frac{11}{20}d$.
6. $\frac{19}{20}a-\frac{5}{12}b+\frac{5}{12}c+\frac{1}{4}d$.
7. $3+9x^2+4x^3-11x^4$.
8. $-x^2-2\frac{29}{35}y^2-z^2+\frac{2}{3}y^3$.

6. $2(a^6+b^6)$. 7. $96a$. 8. $4p^3q^4$. 9. $4xy^2(x^2+y^2)$.
10. $8ab(a^2-b^2)$. 11. $4x(z-y)$. 12. $4x^3y^3$.

EXERCISE 11

9. a^4-b^4. 10. p^4-1. 11. $x^2-(y-z)^2$.
12. $(x+z)^2-y^2$ 13. $p^4+p^2q^2+q^4$. 14. a^8+a^4+1.
15. $(x^2-y^2)^2$. 16. a^2+ab+b^2. 17. $(a^4-1)^2$.
18. $x^4+81+9x^2$. 19 $(a+b-c)(a-b+c)$
20. $(a+b+c+d)(a+b-c-d)$. 21 $(5a+3)(5a-3)$.
22. $(2ac+3b)(2ac-3b)$. 23. $(7c+8d)(7c-8d)$.
24. $(a^2+9b^2)(a+3b)(a-3b)$. 25. $4a(b-c)$.
26. $(4p+7q)(4p-7q)$. 27. $(x+2y-3z)(x-2y+3z)$
28. $(9a-3c)(-3c-a)$. 29. $(3x+4y+7z)(3x+4y-7z)$.
30 $(6a+2b)(12b-2a)$. 31. $(7p+2q)(p-12q)$.
32 0. 33. $(4a-4b+4c)(2c-2a)$.
34. $(6x-2z)(6y-6z-2x)$. 35. $(4p-7q+5r)(2p-q-3r)$.
36. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$. 37. 0
38 0 39 986820. 40. 13732

EXERCISE 12.

13 217 14 504 15. 18 16. 140. 17. 125.
18. 8. 19. 27. 20 125
21. $3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$
22. $3(xy^2-x^2y+yz^2-y^2z+zx^2-z^2x)$ 23. $125(a-b)^3$.
24. $8(a^2+b^2)$. 25. $8c^3$.

EXERCISE 13

1. x^3+8. 2. x^3+27 3 $8a^3+1$. 4. $a^3b^3+c^3$
5 $125x^3+8y^3$. 6. $64a^3+343b^3$. 7. $1-27x^3$.
8 $a^3-8b^3c^3$. 9. x^3-8. 10. $m^3-64p^3n^3$
11. $8a^3b^6-1$. 12. $a^{12}-b^{12}$. 13. $(x+2)(x^2-2x+4)$
14. $(p+4)(p^2-4p+16)$. 15. $(3bc+a)(9b^2c^2-3abc+a^2)$
16 $(2a+7b)(4a^2-14ab+49b^2.)$
17. $(5x-1)(25x^2+5x+1)$. 18. $(1-7x)(1+7x+49x^2)$
19. $(6bc-a)(36b^2c^2+6abc+a^2)$.
20. $(4a-3bd)(16a^2+12abd+9b^2d^2)$.

EXERCISE 14.

1. $x^2+12x+35$. 2. x^2-x-6. 3. x^4-8x^2+15.
4. $x^2+xy-12y^2$. 5 $4x^2+4x-15$. 6. $a^2x^2-ax-156$.
7. $x^3+9x^2+26x+24$. 8. $x^3-6x^2-37x+210$.
9. $8x^3-26x+12$. 10. x^4-10x^2+9.
11. $x^3+\frac{13}{12}x^2+\frac{3}{8}x+\frac{1}{24}$.
12. $1+a+b+c+d+ab+ac+bc+ad+bd+cd+abc+abd$
$+bcd+acd+abcd$.
13. $(a+b)(b+c)(c+a)-abc-a^3-b^3-c^3$.
14. $2(a+b+c)^3+(a+b+c)(ab+ac+bc)+abc$.
15. $x^3+x(ab-ac-bc-a^2-b^2-c^2)+(a-b)(b-c)(c-a)$.
16. $-2(x+y+z)^3+9(x+y+z)(xy+yz+zx)-27xyz$.

EXERCISE 15.

1. $x^6+y^6+z^6-3x^2y^2z^2$. 2. $a^3-b^3-c^3-3abc$.
3. $8x^3-27y^3-z^3-18xyz$. 4. $27p^3-125q^3-64-180pq$.
5. $m^3-n^3+1+3mn$. 6. $x^3+\frac{1}{x^3}-2$.
7. $2(a^3+b^3+c^3-3abc)$. 8 $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3$.

EXERCISE 16.

1. $a^2+b^2+3ab+a^2b+ab^2+a+b$.
2. $\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)+1$.
3. $(x+2y+z)(2x+y+z)(x+y+2z)+(x+y)(y+z)(z+x)$.
4. See Q. 16, Ex. 14. 5. 0. 6. 0.
7. $x+y+x^2(y+1)+y^2(x+1)+2xy$.
8. $2(b+c)^3+bc(b+c)$.
12. $(a-2b+c)(b-2c+a)(2a-b-c)$.

EXERCISE 17.

1. x. 2. $-ac$. 3. $-\frac{a^2b^2x}{3}$. 4. $-\frac{c}{2ab}$.
5. $\frac{a}{b-c}$. 6. $\frac{a}{bxy^2}$. 7. $\frac{x^m}{y^p}$. 8. $\frac{y^{2n}z^n}{x^{2n}}$. 9. 1.

EXERCISE 18.

1. x^2+2x+3. 2. $-x^5+x-2$. 3. $1-a^2$.
4. $c+b+a$. 5. $-\frac{1}{3b}-a-3\frac{b}{a}$. 6. x^2+x+1. 7. $x+2$.
8. $x-2$. 9. $3x-2$. 10. $x-1$. 11. $3x+2$.
12. $(x+a)(x^2+ax+a^2)$. 13. $x^4+a^4+3a^2x^2+2ax^3+2a^3x$.
14. x^2+4x+8. 15. $x-4$.
16. $x^6-3x^5+9x^4-27x^3+81x^2-243x+729$.
17. $x^5+2x^4+x^3-4x^2-11x-10$. 18. $1+2x+3x^2+4x^3$.
19. $3x^3-4x^2y+5xy^2+2y^3$. 20. $3x^3-2x^2-5x-3$.
21. $1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9$.
22. $x^2-(a-b)x-ab$. 23. $(x-1)a+x^2$.
24. $(x^2+xy+y^2)a+x+y$. 25. $c-b$. 26. $ab+ac+bc$.
27. $ab+bc+ac$. 28. $a+b+c$. 29. $x^2+y^2+1-xy+x+y$.
30. $a^2+4b^2+9c^2+2ab+3ac-6bc$.
31. $x^2+y^2+z^2+xy-xz+yz$. 32. $2a-3b-c$.
33. $c^2+ab+ac+bc$. 34. $a^2+ab+ac+bc$.
35. $b^2+ab+bc+ac$. 36. $a+b+c$. 37. $ab+c^2-ac-bc$.
38. $a^2+bc-ab-ac$. 39. $ab-b^2-ac+bc$.
40. $a^2(b-c)+b^2(c-a)+c^2(a-b)$. 41. $ab-b^2-ac+bc$.
42. $a^2(b-c)+b^2(c-a)+c^2(a-b)$. 43. $x^2+y^2+z^2$.
44. $2ab-a^2-b^2+c^2$. 45. $x^2-(a+b)x+ab$.
46. $x^4+(a-b)x^3-(ab+1)x^2-(a-b)x+ab$.
47. $(b+c)x^2+(c+a)x+a+b$. 48. $a^4-3a^2b^2+2b^4$.
49. y^3+y^2+y+1. 50. $x^2-1+\frac{1}{x^2}+3\left(x-\frac{1}{x}\right)+4$.
51. $16(x^4-x^2y^2+y^4)-8a^2(x^2+y^2)+a^4$.

EXERCISE 19.

1. $a+b+c$. 2. $-ap^3+bp^2-cp+d$.
3. $\mp pa^5+qa^4\mp ra+4$. 4. $16a\pm 8b+4c\pm 2d+e$.
5. $\pm mb^5+nb^4\pm pb^3+qb^2\pm rb+s$. 6. 5. 7. 4.
8. -4. 9. -3. 10. a; $3a$.
11. 12. 21. $x^6+x^4y^2+x^2y^4+y^6$.
22. $x^{12}-x^9y^3+x^6y^6-x^3y^9+y^{12}$.
23. $x^{12}-x^{10}y^2+\ldots-x^2y^{10}+y^{12}$.

24. $x^{74}-x^{37}y^{37}+y^{74}$.

25. $x^{18}-x^{16}y^2+\ldots\ldots+x^2y^{16}-y^{18}$.

26. $x^{15}-x^{12}y^3+\ldots\ldots+x^3y^{12}-y^{15}$.

EXERCISE 20.

1. $(x^2+x+1)(x^2-x+1)$. 2. $(x^4-x^2+1)(x^4+x^2+1)$

3. $(x^2+xy+y^2)(x^2-xy+y^2)$. 4. $(a^4+9)(a^2+3)(a^2-3)$.

5. $(x^8+16)(x^2+2+2x)(x^2+2-2x)(x^2+2)(x^2-2)$.

6. $(x^4-x^2y^2+y^4)(x^4+x^2y^2+y^4)$.

7. $(x^2+8+4x)(x^2+8-4x)$.

8. $(2a^2+9+6a)(2a^2+9-6a)$.

9. $(a^2+2b^2+2ab)(a^2+2b^2-2ab)$.

10. $(x^2+3+2x)(x^2+3-2x)$. 11. $(x^2-3+x)(x^2-3-x)$.

12. $(2a^2+3+3a)(2a^2+3-3a)$.

13. $(3x^2-4+3x)(3x^2-4-3x)$.

14. $(x^2+12+4x)(x^2+12-4x)$.

15. $(3x^2+4+5x)(3x^2+4-5x)$.

16. $(3a^2+5x^2+7ax)(3a^2+5x^2-7ax)$.

17. $(3x^2-4+x)(3x^2+4-x)$.

18. $(2b^2+25+10b)(2b^2+25-10b)$.

19. $(x-2y+3z)(x-2y-3z)$.

20. $(a+2b+c)(a+2b-c)(c+a-2b)(c-a+2b)$.

21. $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$.

22. $(a-c)(a+c-2b)$. 23. $(a-d+b-c)(a-d-b+c)$.

24. $(a^2+c^2+b^2-2ac)(a+b+c)(a+c-b)$.

EXERCISE 21.

1. $(x-2)(x^2+2x+4)$. 2. $(a+3)(a^2-3a+9)$.

3. $x^2y(a+3y)(a^2-3ay+9y^2)$.

4. $(x-y)(x^3+x^2y+xy^2-y^3)$.

5. $(x+y)(x^3-3x^2y+3xy^2)$.

6. $(a^2+bc)\ \{(a^2-bc)^2-2bc(a^2-bc)+4b^2c^2\}$.

7. $(x-1)(x^2+x+1)(x^3+2)$. 8. $(a+1)(a^2-4a+7)$.

9. $(a-c)\ \{(a+b)^2+(a+b)(b+c)+(b+c)^2\}$.

10. $3(3a-5)(9a^2+15a+25)$. 11. $(a-2)(a^2+5a+13)$.

12. $(2a+1)(a^2+a+1)$.

13. $(y+z)\{(x+y+z)^2+x(x+y+z)+x^2\}$.

EXERCISE 22.

1. $(x+3)(x-2)$. **2.** $(x+9)(x+16)$.
3. $(x-5)(x-1)$. **4.** $(x-6y)(x-2y)$.
5. $(7x+6y)(7x+y)$. **6.** $(x+1)(14x+1)$.
7. $(x+a)(x+2b)$. **8.** $(x-a)(x-a-b)$.
9. $(x-b+a)(x-b-a)$.
10. $\{x-(a+b)^2\}\{x-(a-b)^2\}$.
11. $(x-a)(x-b-c)$. **12.** $\left(x+\frac{a}{b}\right)\left(x+\frac{b}{a}\right)$.
13. $(x+a+b-c)(x+a-b+c)$. **14.** $3(3x+5)x$.
15. $(x-ab+bc)(x+ab+ac)$. **16.** $(x-c)(x+a+b)$.
17. $(x+7)(x-2)(x+3)(x+2)$. **18.** $(4x+25)(5x+24)$.
19. $(x+1)(2x-1)$. **20.** $(3x-2)(2x-3)$.
21. $(2x-3y)(x+4y)$. **22.** $(3x-2a^2)(2x-5a^2)$.
23. $x^2(6a+x)(4a-3x)$. **24.** $(a^2-3a+1)(a+1)(a-4)$.
25. $(x^2+2x-2)(x+1)^2$. **26.** $(m+6n)(m-5n)$.
27. $(a+4b)(a-3b)$ **28** $(x^2+7)(x+2)(x-2)$.
29. $(a^3-2)(a-2)(a^2+2a+1)$.
30. $(a^4+5)(a^2+4)(a-2)(a+2)$.
31. $(a-1)(a^2+a+1)(a+2)(a^2-2a+4)$.
32. $(2x-5)(x+3)$. **33.** $(3a-5)(2a+3)$.
34. $(4x+3)(2x-3)$. **35.** $(5a-3)(2a-7)$.
36. $(4m+5n)(3m-4n)$. **37.** $(x+a^2+ab)(x+b^2+ab)$.
38. $(x+a^2-ab)(x+b^2-ab)$. **39.** $(x-a+2b)(x-b+2a)$.
40. $(x-2z)(x+y-z)$.

EXERCISE 23.

1. $(a+b)(c-d)$. **2.** $(ax+y)(bx-y)$. **3.** $2(x+2a)(x+3b)$.
4 $(2a-b)(a^2+2bx)$. **5.** $(xy+a^2)(2x-3y)$.
6. $(x+1)(ax^2+bx+a)$. **7.** $(1-x)(1-y)$. **8.** $(x+1)(a+1)$.
9. $(a-b)(a-c)$. **10.** $(p+q)(p+r)$. **11.** $(x-y)(4+b)$.
12. $(a+b)(a+1)$.

EXERCISE 24.

1. $(x^2+ab)(3a-4b)$. **2.** $(x+z)(x-z)(x^2+y^2+z^2)$.
3. $(x+y-z)(x-y+z+1)$. **4.** $(x-y)(x-y-1)$.
5. $(a-b)(a+b+c)$. **6.** $(x+a)^3(x-a)$.

7. $(x+y)(x-y)^3$. 8. $(b-c)(x+a)$.
9. $(a-b-c)(a+b+c+1)$. 10. $(a+b)(a^2+b^2)$.
11. $(a^2+b^2)(a-2b)$. 12. $(y+xz)(x+yz)$.
13. $(a+b)(c+a-b)(c-a+b)$. 14. $(x-y)(1+x)(1+y)$.
15. $(a+b)(a+b+1)$. 16. $(b+c)(a+b-c)(a-b+c)$.

EXERCISE 25.

1. $(b+c-a)(b^2+c^2+a^2-bc+ab+ac)$.
2. $(c+1)(c^2-c+37)$. 3. $(c+4b)(c^2+7b^2-4bc)$.
4. $(3a-b)(3a^2+b^2+3ab)$.
5. $(a-b-c)(a^2+b^2+c^2+ab+ac-bc)$.
6. $(x^2+y^2-z^2)(x^4+y^4+z^4-x^2y^2+x^2z^2+y^2z^2)$.
7. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{c}-\frac{c}{b}-\frac{b}{a}\right)$.
8. $2(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.
9. $\left(x+\frac{1}{x}+1\right)\left(x^2+\frac{1}{x^2}-x-\frac{1}{x}\right)$.
10. $\left(p^3-\frac{1}{27p^3}-2\right)=\left(p-\frac{1}{3p}-1\right)\left(p^2+\frac{1}{9p^2}+p-\frac{1}{3p}+\frac{4}{3}\right)$.

EXERCISE 26.

1. $(a-1)(3a^2+3a+1)$. 2. $(x-1)(2x^2-3x+4)$.
3. $(x-1)(x+1)(2x+1)$. 4. $(x-1)(3x^3-4x^2+5x-7)$.
5. $(x-1)^3(x^2+3x+1)$. 6. $(x-1)^3$.
7. $(a-1)(3a^2-2a+1)$. 8. $(a-1)(a+2)^2$.
9. $(x+1)(3x^3-5x^2+8x-8)$. 10. $(a-1)(4a^2-2a+3)$.
11. $(x+1)(11x^2-4x-1)$. 12. $(x+1)(7x^3-3x^2+x-2)$.
13. $(1-a)(2+a+a^2)$. 14. $(c+1)(c^2-c+37)$.
15. $(x-1)(x^4+1)$. 16. $(x-1)(ax^2+ax+a+bx+b+c)$.
17. $(x+1)^2(x^2-3x+5)$. 18. $(x-1)(2x-1)(3x+1)$.
19. $(x+1)(x^2+2x+3)$. 20. $(x-1)(x^2+2x+3)$.

EXERCISE 27.

1. $3(a+2b+c)(2a+b+c)(a+b+2c)$.
2. $3(2+a+b)(2+b+c)(2+c+a)$.
3. $3(a^2+b^2-ac-bc)(b^2+c^2-ac-ab)(c^2+a^2-ab-bc)$.
4. $3(2-x-y)(2-y-z)(2-x-z)$. 5. $(a-b)(b-c)(a-c)$.

6. $(a-b)(b-c)(c-a)$. **7.** $(a+b+c)(a-b)(b-c)(a-c)$.

8. Same as *Q.* 7. **9.** $(ab+ac+bc)(a-b)(b-c)(a-c)$.

10. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+ac+bc)$.

11. $(a^2-b^2)(b^2-c^2)(a^2-c^2)$.

12. $(a-b)(b-c)(c-a)(a+b+c)$.

13. $(a^2-b^2)(b^2-c^2)(c^2-a^2)$. **14.** Same as *Q.* 6.

15. $(b^2-1)(b^2-c^2)(c^2-1)$.

16. $\left(\frac{a}{b}-\frac{b}{c}\right)\left(\frac{b}{c}-\frac{c}{a}\right)\left(\frac{a}{b}-\frac{c}{a}\right)$.

17. $(a-b)(b-c)(a-c)(a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2$
$+c^2a+ca^2+abc)$.

18. Same as *Q.* 12. **19.** Same as *Q.* 11.

20. $(a^2+b^2)(b^2+c^2)(c^2+a^2)$.

21. $(x^2+y^2)(y^2+z^2)(z^2+x^2)$

22. $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{a}+\frac{a}{b}\right)$.

23. $\left(x^3+\frac{1}{x}\right)\left(x^3+\frac{1}{x^2}\right)\left(\frac{1}{x}+\frac{1}{x^2}\right)$. **24.** $8abc$.

25. $3abc$.

EXERCISE 28.

1. $(2x+3y+4z)(3x+6y+7z)$.

2. $(3a-b+2c)(2a+5b-3c)$. **3.** $(x+2y+3z)(2x+3y+z)$.

4. $(5a+6b+7c)(2a-4b-5c)$. **5.** $(a+2b+c)(a-b+c)$.

6. $(a+b+c)(a+b+c)$. **7.** $(2x+y-z)(x+2y+z)$.

8. $(2a+b+c)(a+2b+c)$. **9.** $(2a-b+c)(a+b+2c)$

10. $(3x-2z-3y)(x-2z-y)$. **11.** $(2x+3a-4)(x-6a+7)$.

12. $(3x+4y+5)(x+y-3)$. **13.** $(3a-2b+4)(2a+5b-7)$

14. $(3x+3y-5)(x-2y+10)$. **15.** $(2a+b+3)(a+2b-3)$.

16. $(7x+6y+2)(2x+3y+6)$. **17.** $(3x+y+3z)(x-2y-2z)$.

18. $(2x+3y-z)(x-y+z)$. **19.** $(2x+3y-z)(x-4y-4z)$.

20. $(a+b+c-d)(a+b+c-d)$.

EXERCISE 29.

1. $3(a^2-b^2)(b^2-c^2)(c^2-a^2)$.

2. $3(a+b-2c)(b+c-2a)(c+a-2b)$.

3. $3abc(a-b)(b-c)(c-a)$.

4. $3(x-a)(x-b)(x-c)(a-b)(b-c)(c-a)$.
5. $3(x+a)(x-b)(a+b)$. 6. $-3c(s-a)(s-b)$.
7. $3abc\left(\frac{b}{c}-\frac{c}{b}\right)\left(\frac{c}{a}-\frac{a}{c}\right)\left(\frac{a}{b}-\frac{b}{a}\right)$.
8. $3a^3b^3c^3(a-b)(b-c)(c-a)$. 9. $3(2a-b)(2b-c)(2c-a)$.
10. $3(x-2y)(2y-1)(1-x)$. 11. $3(ax-by)(by-cz)(cz-ax)$.
12. $3(2a-b-c)(2b-a-c)(2c-a-b)$.
13. $3abc(bz-cy)(cx-az)(ay-bx)$.

EXAMINATION PAPERS-FIRST SERIES.

I.

1. $6x$. 2. $ab+a^2+a^2b^2-2a^3b+b^2+2ab^3$.
3. (i) $14x^4-3x^3+x^2+x-45$.
(ii) $\frac{2}{5}a^4-\frac{136}{75}a^3x+\frac{3}{5}a^2x^2-\frac{3}{1}ax^3+x^4$. 5. $0;0$.
6. $x^2-(a+b)x+ab;\ x-a$. 7. $q-ap+b$.
8. $(x-10)(x+7)$; $(x+1)(x^2-x+2)$.
9. $x^7+x^6a+x^5a^2+\ldots+a^7$. 10. $1+x+x^2+x^3+x^4+x^5$.

II.

1. 2. 2. 3. 3. $2a+16b+15c+10d$; 57.
4. x^4-x^2-2x-1; a^8-16.
5. $x^2+xy+y^2+(p+q+r)(x+y)+pq+qr+rp$.
6. $(p\div 4)$ is an odd integer.
7. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$; $(x-y)(y-z)(x-z)$;
$(x^2+b^2-c^2)(x^2+a^2-b^2)$.

III.

2. $-5(a^2+b^2+c^2)+8(ab+bc+ac)$.
3. (i) $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}$. 3.
4. $a^2+b^2+c^2+ab+bc-ac$; $3a^2-2a+5$. 5. 1.
9. $(b-c)(a-b)(c-a)(a+b+c)$; $3(a-b)(c-b)(a+c)$.
10. $p^3-p^2(a+b+c)+p(ab+ac+bc)-abc$.

IV.

2. $-132.$ **3.** $5x^2-3x+3.$

5. $(a+b)x^3+(b+c)x^2+a+c.$ **6.** $20.$ **7.** $0, 0$

8. $1-2x+2x^2-2x^3+2x^4.$

9. (i) $\{a^2+a(b-c)+(b-c)^2\}(a-b+c).$
(ii) $(x-1)(10x+3).$ (iii) $(a^2-b^2)(b^2-c^2)(a^2-c^2).$

V.

3. $x^2+4y^2+9z^2-2xy+3xz+6yz.$

4. $(x+2y)^2-(x+2y)(3x+z)+(3x+z)^2.$ **5.** $0.$

6. (i) $(a-b)(x+a-b)(x-a+b).$
(ii) $(x+1)(x+2)(x+3)(x+4).$ (iii) $(x+y)(y+z)(z+x).$

8. $a^4-(b^2-2ac)^2.$

VI.

2. $16x+11y.$ **3.** $(bx+cy+az)^2-(cx+ay+bz)^2.$

5. $-1.$ **6.** $\frac{1}{4}(a+b+c)^3.$

9. (i) $(ax-b)(x+1).$ (ii) $(2bx-ay)(2ax-by).$
(iii) $(ax+b)(bx-a).$

VII.

3. $-a^2.$ **4.** 3; $a+b+c-d.$

5. $a^5+3a^4x+9a^3x^2+27a^2x^3+81ax^4+243x^5.$ **7.** $0.$

8. (i) $\left(2+\frac{1}{x^2}+\frac{2}{x}\right)\left(2+\frac{1}{x^2}-\frac{2}{x}\right).$
(ii) $(2a+b-c)(4a^2+b^2+c^2-2ab+bc+2ac).$
(iii) $(a+b+c)(b+c-a)(a-b+c)(a+b-c).$

10. $2mn+2ab.$

VIII.

2. $2(a-b)(b-c)(c-a).$ **3.** $2.$ **4.** $0.$ **7.** $-3(c-a).$

8. $25.$ **9.** (i) $(x+1)(x^2-x+1)(x^2+1)(x^4-x^2+1).$
(ii) $(2a+b+2c)(2a+2b-3c).$
(iii) $2(x+1)(y+1)(xy-1)(x-y).$
(iv) $(x^2+3-2x)(x^2+3+2x).$
(v) $3(a+d+b+e)(a+d+c+f)(b+e+c+f).$

10. $5(a+b)^3.$

IX.

3. $3\sqrt{133}$. 5. $x^3+(a+b)x^2+2abx+1$.

6. $2(x^3+y^3+z^3)+6xy(x+y)+6z(x-y)^2$.

7. (i) $(a+b-1)(a^2+b^2+1+a+b-ab)$.

(ii) $(a^2-1)(b^2-1)$. (iii) $(a+b)3b$.

(iv) $(a+b-3c)(a-b+3c)$. (v) $(x+1)^2(x-1)$.

(vi) $(x-a)(x+a+2y)$. (vii) $(7x-1)(2x-5)$.

10. -1.

X.

2. $a^3+12a^2+48a+64$. 3. 1. 4. $3a^3-8a^2b-4ab^2+3b^3$.

5. $x^6+x^4y^2+x^2y^4+y^6$; $x^{12}-x^{10}y^2+\dots+y^{12}$; $x^6-x^3y^3+y^6$.

7. (i) $(4x^2+3+9x)(4x^2+3-9x)$.

(ii) $\left(2m-\frac{3}{2m}-1\right)\left(4m^2+\frac{9}{4m^2}+2m+4-\frac{3}{2m}\right)$.

(iii) $(x^2+3x+1)^2$

EXERCISE 31.

1. b^2x^2. 2. $5x^2$. 3. $3ab^2$. 4. $3p^2q^3$. 5. x^2y^2.

6. $17a^4b^2c^3$. 7. $9a^2c^3$. 8. $14a^mb^n$.

EXERCISE 32

1. a^2-ab+b^2. 2. $a+1$. 3. $x+1$. 4. $x-2$.

5. $x-b$. 6. $x-y$. 7. x^2+2x+2. 8. $x-1$.

9. $2x+3$. 10. $a-2$. 11. $a+c$. 12. $x+y+z$.

EXERCISE 33.

1. x^2+2x+3. 2. $x-1$. 3. x^2+2x+3. 4. $3x-7$.

5. $2x-1$. 6. $(x+1)^3$. 7. x^2+2x+3. 8. x^2+4x+5.

9. $(x-1)^2$. 10. $(x-a)^2a^2x$. 11. x^2-1. 12. $5x^2-1$.

13. $x-1$. 14. x^2-x+1. 15. $x-1$. 16. $x-1$.

17. $1+a+a^2$. 18. x^2-3x-4. 19. $2x+3a$.

20. x^2+2x+3. 21. $2(x^2-2ax+2a^2)$. 22. $x^2-(a-b)x+b^2$.

23. $x+a$. 24. $x-\frac{\sqrt{x}+1}{2}$. 25. $a+b$. 26. $x-2$.

27. $x+7$. 28. $2x+1$.

EXERCISE 34.

3. $\dfrac{s-q}{r-p}$. **7.** $ab=a^2-c^2$. **13.** $x=4$; $a=4$. **14.** 7. **15.** 8.

EXERCISE 35.

1. $54x^4y^4z^4a$. **2.** $120a^3b^3c^3$. **3.** $b^2x^2(x^2-b^2)$.
4. $(x^2-2x-15)(x+1)$. **5** $(x^4-a^4)(x^2+ax+a^2)$.
6. $(x+1)(x-2)(x+3)(x+4)$. **7.** $(x+a)(x+b)(x+c)$.
8. $(x-b^2)(x-c^2)(x-a^2)$. **9.** $(4x^4-5x^2+1)$.
10. $(a-b)^3(a+b)^3(a^2+ab+b^2)^2(a^2+b^2)$.
11. $3(a-b)(b-c)(c-a)$. **12.** $(a^6-b^6)(b+c)(c+a)$.

EXERCISE 36.

1. $(3a^2-a+1)(2a-3)(3a-2)$.
2. $(x-4)(x-5)(x^2-5x+6)$.
3. $(x+c)\{2x^3+x^2(2a-3b)-x(2b^2+3ab)+3b^2\}$.
4. $(x+1)(2x+1)(3x^2-x+2)(2x^2-x-4)$.
5. $(a^2+2a+3)(3a^2+8a)(4a^2+9)$.
6. $(3x-1)(2x-3)(2x+1)(2x+9)$.
7. $(3a-2)(4a-1)(2a+3)(2a-5)$.
8. $(a^4-3a+20)(3a+4)(a+3)$.
9. $(x^5-2x^4-x-1)(x^3+x^2+2x+1)$.
10. $(2a+3b+c)(a-b-c)(a-4b+4c)$.
11. $x^3+2ax^2-a^2x-2a^3$. **16.** -2.
17. $(x-5)(x-6)$ and $(x-2)(x-5)$.
18. $(x+4)(x-3)$ and $(x+4)(x-5)$.
19. $(x+1)(x+2)$ and $(x+1)(x+3)$.

EXERCISE 37.

1. $\dfrac{x-a}{x}$. **2.** $-\dfrac{1}{3+a}$. **3.** $\dfrac{2a-3b}{2a}$. **4.** $\dfrac{3a}{b+4a}$.

5. $a+\dfrac{b^2}{a+b}$. **6.** $\dfrac{1}{x^2-y^2}$. **7.** $\dfrac{x-1}{x-4}$. **8.** $\dfrac{3x+4}{2x^2+5x-1}$.

9. $\dfrac{x-4}{x-5}$. **10.** $x-3$. **11.** $a+b$. **12.** $\dfrac{4x+2}{x+1}$.

13. $\dfrac{3x-1}{2x-1}$. **14.** $\dfrac{x-y}{x^2+y^2}$. **15.** $\dfrac{x-2}{2x-1}$.

16. $\dfrac{x^4+a^4}{x^4+x^3a+x^2a^2+xa^3+a^4}$. 17. $\dfrac{x-10}{x^2-7x+10}$.

18. $\dfrac{x^2-12x+35}{3x-17}$. 19. $\dfrac{5x(5x^2+1)}{9x^2-4}$. 20. $\dfrac{x^2+2xy+4y^2}{2(2x-3y)}$.

21. $\dfrac{3ax^2+1}{4a^2x^4+3ax^2-1}$. 22. $\dfrac{5a^3(a+x)}{x^2(x^2+ax+a^2)}$.

23. $(b+c-a)(c+a-b)$. 24. $\frac{1}{2}(a+b+c)$. 25. -3.

26. $\dfrac{1}{b+c}$. 27. $3(a-b)$. 28. 3. 29. $\dfrac{(a-1)(b-1)}{(a+1)(b+1)}$.

30. $\dfrac{a-b+c}{a-4b-4c}$.

EXERCISE 37-A.

1. $\dfrac{2aby^2,\ 3b^2yx,\ z}{6b^2y^2}$. 2. $\dfrac{bc,\ ac,\ ab}{abc}$. 3. $\dfrac{a^2,\ b^2,\ c^2}{abc}$.

4. $\dfrac{c,\ a,\ b}{abc}$. 5. $\dfrac{(b-c)(c-a),\ (c-a)(a-b),\ (a-b)(b-c)}{(a-b)(b-c)(c-a)}$.

6. $\dfrac{c-a,\ a-b,\ c-b}{(a-b)(b-c)(c-a)}$. 7. $\dfrac{a(a-b),\ b(a+b),\ ab}{ab(a^2-b^2)}$.

8. $\dfrac{(a-b)(a^3+b^3),\ (a+b)(a^3-b^3),\ 1}{a^6-b^6}$.

9. $\dfrac{x-3,\ x-1,\ x-2}{(x-1)(x-2)(x-3)}$.

10. $\dfrac{(x^2+x+1)(x+1),\ (x-1)(x^2+1),\ x^2}{x^3-1}$.

11. $\dfrac{a(b+c),\ b(c+a),\ c(a+b)}{(a+b)(b+c)(c+a)}$.

12. $\dfrac{(a-b)(x-c),\ (b-c)(x-a),\ (c-a)(x-b)}{(x-a)(x-b)(x-c)}$.

EXERCISE 38.

1. $\dfrac{2}{(1+x)^2-x^4}$. 2. $\dfrac{2(4a^2+9b^2)}{4a^2-9b^2}$.

3. $\dfrac{8x+5}{(2x+1)(4x+3)(3x+2)}$. 4. $\dfrac{2ab}{a^2-b^2}$. 5. $\dfrac{4a+x}{a-x}$.

6. $\dfrac{2}{x(x+1)(x+2)}$. 7. $\dfrac{3x^2}{x^3-1}$. 8. $\dfrac{4a^3}{a^4-x^4}$.

9. $\dfrac{2x-3}{(x^2-1)(2x+3)}$. **10.** $\dfrac{2}{x(1-4x^2)}$. **11.** $\dfrac{ab(3a-2b)}{(a-b)^3}$.

12. 0. **13.** $\dfrac{4ab}{(a-b)^2}$.

14. $\dfrac{bx^3+bc(a^2-ax+x^2)+a(a^3+x^3)}{b(a^3+x^3)}$. **15.** 1.

16. $\dfrac{4x-3y}{4x^2-y^2}$. **17.** $\dfrac{1}{(x+1)(x+3)}$. **18.** $\dfrac{x(x^2-2x+2)}{(x-1)^2}$.

19. $\dfrac{x}{(1-x)(1+x^3)}$. **20.** $\dfrac{a^2}{(a-2)(a+1)^2}$.

21. $\dfrac{12}{(x^2-1)(x^2-4)}$. **22.** $\dfrac{2(x+8)}{(1-x)(x+5)}$. **23.** 0.

24. $\dfrac{(a-b)(b-c)(a-c)}{(a+b)(b+c)(c+a)}$. **25.** $\dfrac{3x-11}{(x-1)(x-3)(x-5)}$.

26. 0. **27.** 3. **28.** $\dfrac{x^2}{(x-a)^3}$. **29.** $\dfrac{2a^2-31a-2}{a^4-4}$.

30. $\dfrac{2}{(1-a^2)(1-a^4)}$.

EXERCISE 39.

1. $b(a^2+b^2)$. **2.** $a+b$. **3.** $\dfrac{(x-4)(x-7)}{x^2}$. **4.** $\dfrac{4ab}{a^2-b^2}$.

5. $\dfrac{a^2(a-b)}{x}$. **6.** $\dfrac{x^2}{a^2-x^2}$. **7.** 1. **8.** $\dfrac{b^2(a^2-1)}{a^2+1}$.

9. $\dfrac{x^2+y^2}{xy}$. **10.** $\dfrac{x^4-y^4}{x^2y^2}$. **11.** $\dfrac{(a+b+c)^2}{2bc}$.

12. $\dfrac{y(yz+1)}{(xy+1)(xyz+x+z)}$. **13.** $\dfrac{a}{b}$. **14.** $\dfrac{1}{a^2+b^2}$.

15. $\dfrac{(a+b-c)(c-a+b)}{4}$. **16.** $\dfrac{(c+d)^2-(a-b)^2}{4(ab+cd)^2}$.

17. $\dfrac{ax(a^2-ax+x^2)}{(a-x)(a^3-x^3)}$. **18.** 1. **19.** $\dfrac{a-c}{ac+1}$.

20. $\dfrac{b^2-a^2+bc}{b^2-a^2-ac}$. **21.** 1; $(a^2-b^2)(a^2+ab+b^2)$.

22. $\dfrac{1}{(x+2)^2(x+1)^2(x-6)}$; $x^2(x^2-2x+4)(x^2-4)$.

23. $\dfrac{a-1}{(a^2+1)(a^3+1)(a+1)}$; $(a^3-1)(a^2-1)$.

24. $\dfrac{1}{abc(a+b)(b+c)(c+}$; $a^3b^3c^3(a-b)(b-c)(c-a)$.

EXERCISE 40

1. 1. **2.** 0. **3.** 1. **4.** $n+m$. **5.** -1. **6.** -1.
7. 1. 1. **9.** $a+b+c$. **10.** $a^2+b^2+c^2+ab+ac+bc$.

11. $\dfrac{1}{(a-1)(b-1)(c-1)}$. **12.** $\dfrac{1}{abc}$. **13.** $\dfrac{1}{(a+x)(b+x)(c+x)}$.

14. $\dfrac{1}{(a-1)(b-1)(c-1)}$. **15.** $\dfrac{1}{(a-1)(b-1)(c-1)}$.

16. $\dfrac{x}{(a-x)(b-x)(c-x)}$. **17.** $\dfrac{x^2}{(a-x)(b-x)(c-x)}$.

18. $\dfrac{a+b+c-ab-ac-bc}{(a-1)(b-1)(c-1)}$. **19.** $\dfrac{(x+1)^2}{(a-x)(b-x)(c-x)}$.

20. $\dfrac{mx^2-nx+p}{(a+x)(b+x)(c+x)}$.

EXERCISE 41.

1 0. **2.** 0. **3.** $\dfrac{2(3x^2-12x+11)}{(x-1)(x-2)(x-3)}$ **4.** 0

5 $\dfrac{2}{a-b}$. **6.** 0. **7.** x. **8.** 2. **9.** $\dfrac{1}{x}$. **10.** -1.

11. 1. **12.** $-n$. **13.** 1 **14.** 1. **15.** $\dfrac{3(x+y)}{x-y}$.

16. $2(a+b+c)$. **17.** $a^2+b^2+c^2$. **18.** $2(a^2+b^2+c^2)$

19. $-3(a+b)(b+c)(c+a)$. **20.** $-\dfrac{3abc}{a+b+c}$. **21.** 3.

22. $(a+b+c)$. **23.** $ab+ac+bc$.

24. $\dfrac{4}{(x+1)(5x+1)}$; $\dfrac{n}{(x+1)\{(n+1)x+1\}}$.

EXERCISE 42.

1. $\dfrac{1}{x-a}-\dfrac{1}{x-b}$. **2.** $\dfrac{2}{x-4}+\dfrac{1}{x+2}$.

3. $\dfrac{1}{2(x+2)}-\dfrac{1}{2(x+6)}$.

4. $\dfrac{1}{x+2}-\dfrac{2}{x^2-2x+4}$.

5. $\dfrac{1}{2x-1}-\dfrac{1}{3x-1}$.

6. $\dfrac{1}{x+a+b}+\dfrac{1}{x-a-b}$.

7. $\dfrac{q-ap}{(b-a)(x+a)}+\dfrac{bp-q}{(b-a)(x+b)}$.

8. $\dfrac{3}{2(a+1)}-\dfrac{3}{a+2}+\dfrac{3}{2(a+3)}$.

9. $\dfrac{ap^2+bp+c}{(p-q)(p-r)(x-p)}+\dfrac{aq^2+bq+c}{(q-p)(q-r)(x-q)}+\dfrac{ar^2+br+c}{(r-p)(r-q)(x-r)}$.

10. $\dfrac{3}{y-1}+\dfrac{2}{y-3}+\dfrac{5}{y-2}$.

11. $\dfrac{1}{x-2}-\dfrac{2}{x-3}+\dfrac{3}{x-4}$.

12. $\dfrac{4}{x+2}+\dfrac{5}{x+3}+\dfrac{3}{x+4}$.

EXERCISE 43.

1. 2. **2.** $\dfrac{3a^2(b^2-a^2)}{b^2(a+2b)^2}$. **3.** 0. **4.** 1. **5.** 0. **6.** 1.

7. $\dfrac{1}{a}$ **8.** 1. **9.** $\dfrac{ab}{a-b}$. **10.** 1. **11.** $-\dfrac{b}{a}$.

12. $\dfrac{c^2+cd+d^2}{c+d}$. **13.** $\dfrac{(a-b)^3}{(a+b)(a^2+b^2)}$. **14.** 0. **15.** 1.

16. $\dfrac{2}{b}$. **17.** 1. **18.** $\dfrac{ab}{a^2+b^2}$. **19.** 1. **20.** a.

EXAMINATION PAPERS—SECOND SERIES.

I.

3. $4x-5$ **4.** $(x-1)(x-2)(x-3)(x-4)$. **7.** $\dfrac{x^2-xy+y^2}{x^2+xy+y^2}$.

8. 0. **10.** $\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}$.

II.

1. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

5. $6x^5-x^4-43x^3+43x^2+x-6$.

7. (i) $(2n+1)^2$ (ii) $\dfrac{x(x^3+x^2+1)}{x^4-1}$. **9.** $\dfrac{4}{3}$.

III.

3. $\dfrac{1}{(a-b)(b-c)(c-a)}$. **4.** x^2-3x+1. **6.** $\dfrac{5x+2}{7x-4}$.

7 $(a+b+c)^2$. **9.** 0. **10.** $p=\frac{1}{2}$; $q=-1$; $r=\frac{1}{2}$.

IV.

1. $(2a-1)(12a^3+4a^2-3a-1)$. **3.** $\dfrac{a^{2m}+2}{a^m+2}$.

5. x^2+xy+y^2. **6.** (i) $2(a-b)(1-ab)$.
(ii) $(x+y)(z+x-y)(z-x+y)$.

10. (i) $\dfrac{x(y^3+2xy)}{y^4+2xy^2+x^2}$. (ii) $\dfrac{x(x+2)}{x^2+3x+1}$.

V.

2. $p-q$. **5.** (i) $(a^2+\dfrac{1}{a^2}+1)(a^2+\dfrac{1}{a^2}+3)$. (ii) $(x+a)^2$
$(x+b)^2$. **6.** (i) $3(a+b+c)$. (ii) $\dfrac{x^4}{(x-a)^n}$. **10.** 1.

VI.

2. $6(4x-y)(3x^3-3x^2y+xy^2-y^3)$. **3.** 6; 30.

5. (i) 4. (ii) $2abxy$. **9** 1.

VII.

1. $(ac+bd)(bc+ad)$. 2. $x+y-3$. 3. $x=$
7. $a^7(a^9+1)(a^6-a^5+a^4-a^3+a^2-a+1)$. 8. a^2+3a+
9. (i) $2(5x^2+5y^2-8xy)$. (ii) $p+q+r$. (iii)

VIII.

6. (i) 0. (ii) 0. 10. $\frac{2}{x+1}+\frac{3}{x+2}-\frac{5}{x+3}$.

IX.

1. $b^4=a^5$. 3. $(a-4)(a-1)$ and $(a\ \ 4)(a-2)(a-3$
4. $p=q=r=1$. 5. $\frac{9x^2-x-3}{4(x+5)}$. 6. (i) -3. (ii)
9. $n(n-1)$.

X.

1. $x-1$. 3. H.C.F. of *Nr.* and *Dr.* is x^2-3x+
4. $24xyz$. 7. (i) $(x+a+b)(x^2+a^2+b^2-ax-bx-al$
(ii) $(x-a-b-c)\{(x-a)^2+b^2+c^2+b(x-a)+c(x-a)-bc$
9. (i) $x+y+z$. (ii) 1. (iii) 2.

EXERCISE 45.

13. $2(6a^5b+20a^3b^3+6ab^5)$.
14. $2(x^7+21x^5y^2+35x^3y^4+y^7)$.
15. $5ab(a+b)(a^2+ab+b^2)$. 16. $7ab(a+b)(a^2+ab+b^2)$
17. $80a$. 18. $-12y$. 19 $-5103a^5$. 20. 16. 21. 6. 22.

EXERCISE 46.

1. $9ab+c^2$. 2. $7a-3b$. 3. ax^2+bx+c. 4. $3-x+2$
5. b^2-2ba^2-a. 6 x^3+4x-1. 7. $\frac{x^2}{2}-2x+\frac{a}{3}$.
8. $3x^2-\frac{1}{3}xy+3y^2$ 9. $x^2-x+\frac{1}{4}$. 10. $\frac{x^2}{2y^2}+1+\frac{2y^2}{x^2}$.
11. $\frac{x}{y}-\frac{y}{2x}-\frac{1}{2}$. 12. $x+\frac{1}{x}-1$. 13. $\frac{1}{a^2}+\frac{1}{a}-1+a$.
14. $a+b$. 15. $a^2-b^2+c^2-d^2$. 16. $a^2+(2b-c)a+c^2$

17. $a-2-\frac{1}{a}$. **18.** $a+\frac{3}{2a-1}$. **19.** $\frac{2x^2+x-2}{2x-1}$.

20. $a+\frac{1}{a+2}$.

EXERCISE 47.

1. $3x^2-2x+1$. **2.** $x^2-x+\frac{1}{4}$. **3.** $x+\frac{a}{3}-\frac{b}{2}$.

4. $(x+y)^3$. **5.** $x^2-3xy+2y^2$. **6.** $x^4-a^2x^2-a^4$.

7. $\frac{3a^2}{b^2}+\frac{4a}{b}+3-\frac{2b}{a}$. **8.** $4a^2+2a-1-\frac{3}{a}$.

9. $2(x-1)^2+5(x-1)-6$. **10.** $(a-1)^2+3(a-1)-2$.

11. $a+b-c$. **12.** $a^2+c^2-2b^2$. **13.** $a-\frac{1}{a}-2$.

14. $a^2+\frac{1}{a^2}+1$. **15.** $x+\frac{1}{x}+2$. **16.** $\frac{x}{y}+\frac{y}{x}+1$.

17. $x=10$. **18.** $x=2$. **19.** $x=\frac{a^4+b^4}{b^3}$. **20.** $x=3$.

21. $x=6a$. **22.** $x=9\frac{7}{8}$. **23** $2x^2-x+1$. **24.** x^2-3x+3.

25. $2x^2+2x+3$. **26.** a^3+a^2+3a+3.

27. $c-3ab+2a^3=0$; $3b-2a^2=\sqrt{d}$. **28.** $p^2=4q$.

29. $r^2=p^2s$; $p^3+8r=4pq$. **30.** $p-q=3p^2q^2$; $p^3=6q^3$

31. $1-\frac{a^2}{2}-\frac{a^4}{8}-\frac{a^6}{16}-\frac{5a^8}{128}$; $1+\frac{a^2}{2}-\frac{a^4}{8}+\frac{a^6}{16}-\frac{5a^8}{128}$.

32. $a\left(1-\frac{x^2}{2a^2}-\frac{x^4}{8a^4}-\right)$; $a\left(1+\frac{x^2}{2a^2}-\frac{x^4}{8a^4}+\right)$.

33. $1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}+\frac{3}{128}x^4$; $1-\frac{x}{2}+\frac{3}{8}x^2+\frac{3}{16}x^3+\frac{3}{128}x^4$.

34. $a^{\frac{1}{2}}\left(1+\frac{b}{2a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}\right)$;

$a^{\frac{1}{2}}\left(1-\frac{b}{2a}-\frac{b^2}{8a^2}-\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}\right)$.

EXERCISE 48.

1. $2x^2-3x+1$. 2. $a+\frac{1}{a}+1$. 3. $\frac{x^2}{4}-x+1$.

4. x^2-x+2. 5. $4x^2-3x+1$. 6. $a-\frac{1}{a}+1$.

7. $a-4+\frac{2}{a}$. 8. $a^2+\frac{1}{a^2}-2$. 9. $(a-b)^2$.

10. x^2+x+1. 11. $a+\frac{1}{a}-1$. 12. $x-y-z$.

EXERCISE 49.

1. x^2-3x+2. 2. $2x^2+4cx-3c^2$ 3. $(x-1)^2$.

4. x^2+x+1. 5. $x^2+2+\frac{1}{x^2}$. 6 $3x^2-2x+1$.

7. $x=2\frac{1}{2}$. 8. $x=1$. 9. $x=1$. 10. $x=7$. 11. $x=13$.

12 $1-\frac{a}{2}$. 13. $x-\frac{1}{x}$. 14. $x-2$. 15. $3b=a^2$; $c=a^3$.

16. $p^2=3q$; $\sqrt[3]{r}=\frac{p}{3}$. 17. $4c^2=9bd$; $bd=16ae$.

18. $1-\frac{x}{3}-\frac{x^2}{9}-\frac{5x^3}{81}$. 22. $qr=100t=4ps$.

EXERCISE 50

1. $a^{-3}+a^3$. 2. $a-1$. 3. $x^{\frac{1}{4}}(x^{\frac{1}{4}}-y^{\frac{3}{2}})$. 4. x.

5. $a^{\frac{2}{3}}+b+c^{\frac{1}{2}}+a^{\frac{1}{3}}b^{\frac{1}{2}}-a^{\frac{1}{3}}c^{\frac{1}{4}}+b^{\frac{1}{2}}c^{\frac{1}{4}}$. 6 $x^p+x^{\frac{p}{2}}y^{\frac{p}{2}}+y^p$.

7. 1. 8. $a^{\frac{3}{2}}+b+c^{\frac{3}{4}}-3a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$. 9. 0.

10 $(x^{-1}-4)(x^{-1}-3)$; $(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}})$;
$(a^2+b^2+ab\sqrt{2})(a^2+b^2-ab\sqrt{2})$; $(x+x^{-1}+1)$
$\times(x^2+x^{-2}-x-x^{-1})$; $(a^{-1}+a^{-2})(a^{-2}+a^{-3})(a^{-3}+a^{-1})$.

11. $x+y+\sqrt{xy}$; $(x-y)\times$ 2nd Expn. **12.** $\left(\frac{a}{b}\right)^{\frac{p}{q}}$, $=q$.

13. 2^n+1+2^{2n}. **14.** $yx^{-\frac{1}{2}}+y^{\frac{1}{4}}x^{\frac{1}{4}}-\frac{1}{2}xy^{-\frac{1}{2}}$.

15. $2x^{\frac{1}{2}}-3y^{\frac{1}{3}}+4z^{\frac{1}{4}}$. **16.** $x^{\frac{2}{3}}+4x^{\frac{1}{2}}+x-4x^{\frac{7}{12}}-4x^{\frac{3}{4}}+2x^{\frac{5}{6}}$.

17. $\frac{a+x}{x^2+3ax+a^2}$. **18.** $\frac{(1+3x^{\frac{1}{2}})^2}{(1-x^{\frac{1}{2}})^2}$.

EXERCISE 51.

1. $3\sqrt[3]{5}$. **2.** $41\sqrt{2}$. **3.** $-4a$. **4** $-20b\sqrt[4]{2a}$

5. $-\frac{1}{\sqrt[3]{9}}$. **6.** $-3ab$. **7** $\sqrt[3]{3}$, $\sqrt{2}$, $\sqrt[4]{6}$ **8.** $\sqrt[3]{3}$ is greater; $\sqrt[3]{6}$ is greater. **9.** $a\sqrt{b}$, $xy\sqrt[3]{x}$, $(x+1)\sqrt{x-1}$, $\sqrt[3]{\frac{1}{4(x-3)}}$.

10. $\sqrt[8]{x^4}$, $\sqrt[8]{y^2}$; $\sqrt[12]{64}$, $\sqrt[12]{27}$. **11.** $3\sqrt{7}>4\sqrt[3]{3}>4\sqrt{2}$.

12. 42; $\sqrt[6]{2^{13}}$; $288\sqrt{2}$; $3ab\sqrt[3]{2b}$ **13.** $10\sqrt{3}$; $24\sqrt[3]{3}$; 1.

$3\sqrt{3}+\frac{7}{\sqrt{2}}$. **15** $\pm b$ **16.** $2x-2\sqrt{x^2-4a^2}$, $5a^2+5b^2+10ab$; $2x^2-2\sqrt{x^4-4y^4}$; $13a-5b+12\sqrt{a^2-b^2}$.

17. $a+b-c+2\sqrt{ab}$. **18.** $2ab+2bc+2ac-a^2-b^2-c^2$.

19. b. **20.** 23.

EXERCISE 52.

1. $\sqrt{2}+1$. **2.** $16+11\sqrt{2}$. **3.** $\frac{1+\sqrt{1-x^2}}{x}$.

4. $\frac{a+\sqrt{a^2-x^2}}{x}$. **5.** $\frac{1+\sqrt{1-x^4}}{x^2}$. **6.** $\frac{\sqrt{2}+2-\sqrt{6}}{4}$.

7. 13·928; 11·813; 27·435. **8.** 47. **9** 1. **10.** $\frac{2a}{b^2}$.

11. $\sqrt{3}$. **12.** 0. **13.** x **14** 1. **15.** $\frac{1}{4}(m^{\frac{3}{2}}+2m^{\frac{1}{2}}n)$.

EXERCISE 53.

1. $1-2^{\frac{1}{3}}+2^{\frac{2}{3}}$ 2. $a^{\frac{5}{2}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}+ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}+b^{\frac{10}{3}}$.

3. $\{(\sqrt{2})^6-(\sqrt[3]{2})^6\}\div(\sqrt{2}-\sqrt[3]{2})$.

4. $\{(\sqrt{3})^4-(\sqrt[4]{5})^4\}\div(\sqrt{3}+\sqrt[4]{5})$.

5. $\{(\sqrt{2})^4-(\sqrt[4]{3})^4\}\div(\sqrt{2}-\sqrt[4]{3})$.

6. $\{a^4-(\sqrt[4]{b})^4\}\div(a-\sqrt[4]{b})$.

7. $\{(\sqrt{5})^4-(\sqrt[4]{7})^4\}\div(\sqrt{5}+\sqrt[4]{7})$.

8. $\{\sqrt{12})^6-(\sqrt[3]{54})^6\}\div(\sqrt{12}-\sqrt[3]{54})$. 9. 0. 10. 1.

11. $a+b$. 12. $2+\sqrt{3}$. 13. $3-\sqrt{7}$.

14. $6+\sqrt{7}$. 15. $\sqrt{a+b}+\sqrt{a-b}$.

16. $\sqrt{\frac{3}{2}}-\sqrt{\frac{5}{2}}$. 17. $1+\sqrt{3}$. 18. $\sqrt[4]{27}(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})$.

19. $\sqrt{3}(2-\sqrt{21})$. 20. $\sqrt{\dfrac{(a+c)(b+c)}{2}}+\sqrt{\dfrac{(a-c)(b-c)}{2}}$.

21. $\frac{5}{3}\sqrt{3}-2$. 22. $1+\sqrt{2}+\sqrt[3]{3}$. 23. $\sqrt{3}+\sqrt{2}-2$.

24. $\dfrac{\sqrt{2}+\sqrt{5}-\sqrt{3}}{\sqrt{2}}$. 25. $1+\sqrt{3}$. 26. $1+\sqrt{5}$.

27. $\sqrt{3}-\sqrt{2}$ 28. $1-\sqrt{2}$. 29. $2+\sqrt{3}$. 30. $3+\sqrt{2}$.

31. $2+\sqrt{2}$. 32. $1+\sqrt{x}$. 33. 0.

EXERCISE 54.

1. 2. 2. a.

EXAMINATION PAPERS—THIRD SERIES.

I.

1. $-19600x^3y^{47}+1225x^2y^{48}-50xy^{49}+y^{50}$; $+\dfrac{45\times44\times43}{6}$

$\times(3a)^3(2b)^{42}-45\times22(3a)^2(2b)^{43}+45\times3a(2b)^{44}-(2b)^{45}$

$-3300\times49a^3b^{97}+4950a^2b^{98}-100ab^{99}+b^{100}$.

2. $2n$ or $2n-1$ digits ; $3n$, $3n-1$ or $3n-2$ digits.

4. (i) $\sqrt{\frac{1}{2}(n^2+n+1)}+\sqrt{\frac{1}{2}(n^2-n+1)}$.

(ii) $\sqrt{\frac{1}{2}(n^2+2+2n)}-\sqrt{\frac{1}{2}(n^2+2-2n)}$. (iii) $7-\sqrt{3}$.

(iv) 2^n+1. (v) $\sqrt{3}+\sqrt{5}+\sqrt{7}$.

6. (i) $2+\sqrt{2}$. (ii) $a+\sqrt{b}$. (iii) $2a$.

7. (i) $\dfrac{1+2x^2+2x\sqrt{1+x^2}}{2x^2-1}$.

(ii) $(\sqrt[4]{a+b}+\sqrt[4]{a-b})^2(\sqrt{a+b}+\sqrt{a-b})\div 2b$.

8. (i) $(a^{-2}+a^{-1}b^{-1}+b^{-2})(a^{-2}-a^{-1}b^{-1}+b^{-2})$.

(ii) $(x^{\frac{1}{3}}+y^{\frac{1}{3}}-z^{\frac{1}{3}})(x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}z^{\frac{1}{3}}+y^{\frac{1}{3}}z^{\frac{1}{3}})$.

(iii) $(x+y)^{\frac{1}{2}}(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z})$.

10. $(a+b)^{\frac{1}{2}}$.

II.

1. (a) $4a^2b^2$. (b) $(a^2-x^2)^2$. 2. $x-a+2\sqrt{b}$; a^x+1+a^{-x}.

5. $x+\dfrac{1}{x}$. 6. $(x^{\frac{2}{3}}-a^{\frac{2}{3}})(x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}})$; $\sqrt{xy}+1$.

III.

1. $4a^2b^2c^2$. 2. -238 ; $a^4+12a^2b+6b^2$.

3. (i) $a+\dfrac{x}{2a}-\dfrac{3x^2}{8a^3}$. (ii) $a^{2m}-2a^{m+n}+a^{2n}$.

(iii) $\frac{3}{2}x^{\frac{3}{2}}-{}^5xy^{\frac{1}{2}}+\frac{2}{5}x^{\frac{1}{2}}y$. 4. (a) 1.

(b) $\dfrac{\sqrt{3}-1}{(2-\sqrt{3})(\sqrt{3}+1)}$. 7. $\frac{1}{2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$.

8. a^2-a+2. 9. $9a^2-6ab+4b^2$. 10. 1.

IV.

1. $\dfrac{1}{x}$ 2. $(a^2-b^2)^{\frac{1}{6}}$. 3. 4. 4. $x=1$.

7. $-42a+49$; a^2-3a+7. 8. $a=7$.

10. A $=-8$; B $=-12$; C $=20$.

V.

2. $4+\sqrt{3}$. 3. $x^{\frac{4}{5}}-a^{-\frac{3}{5}}x^{\frac{7}{5}}+a^{\frac{4}{5}}$. 4. 1. 5. 2.

8. (i) $\dfrac{y^2}{x^2}$. (ii) $\dfrac{3\sqrt{2}}{5}$. 9. $-60a^2+225a-125$; a^2+3a-5.

10. $9(1+a+a^2)$.

VI.

1. $-2(a^2b^2c+b^2c^2a+c^2a^2b)$. 3. $qr=9ps$; $q^3=27p^2s$.

4. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})$. 5. c. 6. 0.

7. (i) $\sqrt{x}-\sqrt[3]{x}+1$. (ii) $m^{\frac{3}{2}}-m-m^{\frac{1}{2}}-1$.

(iii) $x^{\frac{1}{3}}+x^{\frac{1}{6}}+1-x^{-\frac{1}{6}}$. **8.** 0 ; 1.

VII.

1. (*a*) $1+\sqrt{2}-\sqrt{3}$. (*b*) $2+\sqrt{3}-\sqrt{8}$. (*c*) $\sqrt{3}-2\sqrt{\frac{2}{3}}$.
(*d*) $7+4\sqrt{3}$. **2.** $a+b$. **3.** $a^2d=c^2$; $a^2+8\sqrt{d}=4b$.
6. $(x^2+12ax+31a^2)^2-16a^4$. **7.** 10. **9.** 0.

VIII.

1. $(a^3x^6-1)(a^2x^4-1)$. **2.** $(a^2-b^2)^3$.

3. $a^{\frac{2}{3}}+b^{\frac{4}{3}}+c^2-a^{\frac{1}{3}}b^{\frac{2}{3}}-ca^{\frac{1}{3}}-cb^{\frac{2}{3}}$. **4.** $\frac{1}{2}x^{\frac{5}{3}}-5y^{\frac{4}{5}}$.

5. (*a*) $\dfrac{(x-a^{\frac{1}{2}})(x+a^2)}{x-a}$. (*b*) $\dfrac{2a^{\frac{1}{3}}x-1}{a^{\frac{2}{3}}x^2-a^{\frac{1}{3}}x}$. **6.** 15 ; 52.

IX.

1. $p+q$ or $p+q-1$. **2.** $\dfrac{1}{n}$. **5.** (i) $(1-a^{\frac{1}{2}})(1-a)$.

(ii) $(a^{-\frac{1}{2}}+b^{-\frac{1}{3}}+c^{-1})$
$\times(a^{-1}+b^{-\frac{2}{3}}+c^{-2}-a^{-\frac{1}{2}}b^{-\frac{1}{3}}-a^{-\frac{1}{2}}c^{-1}-b^{-\frac{1}{3}}c^{-1})$.

7. (i) $2x^{\frac{1}{2}}y^{\frac{1}{2}}-x-y+z$. (ii) $(x^{\frac{2}{3}}-x^{\frac{1}{3}}+1)(x^{\frac{4}{3}}-x^{\frac{2}{3}}+1)$.

8. $x-\dfrac{1}{x}$.

X.

1. a to the power of 2^{27} ; a^{18}. **2.** $144a^2$, $-165\times3^{11}a^3$

3. 2 ; $16b^4$. **4.** $a+\dfrac{1}{a}$. **5.** $4c$.

6. $\left(xy+\dfrac{1}{xy}\right)^2+\left(xy-\dfrac{1}{xy}\right)^2$. **8.** 2 **9.** $\dfrac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}}$.

EXERCISE 55.

1. 14. **2.** 3. **3.** 2. **4.** 2. **5.** $3\frac{1}{2}$. **6.** $1\frac{1}{5}$. **7.** 7.
8. 5. **9.** $-1\frac{3}{4}$. **10.** 3. **11.** 7. **12.** -4.
13. $(bc-ab)\div(a+b-c)$. **14.** $ab\div(a+b-c)$. **15.** -3.

16. 12. **17.** 5. **18.** $a \div (1-b+a)$. **19.** 1. **20.** $1\frac{1}{2}$.
21. $10\frac{7}{10}$. **22.** $-\frac{3}{10}$. **23.** $-\frac{1}{3}$. **24.** 1.
25. $(a^2+b^2)\div 2a$. **26.** $(cd-ab)\div(c+d-a-b)$.
27. $(10-2a)\div(3a-3)$. **28.** $m^2 \div n$. **29.** $-2b$.
30. $-2(ab+ac+bc)\div(a+b+c)$. **31.** $-2ab\div(a+b)$.
32. $5a\div 4$. **33.** $(3a^2-7ab)\div(3b-a)$. **34.** $24ab\div(a+7b)$.
35. $-5ab\div 2(a+b)$. **36.** $\frac{1}{3}(a+b+c)$. **37.** $c+d$.
38. $-\frac{1}{3}(a+b+c)$ **39.** $(10a^2+b^2)\div 8(a+2b)$.
40. $ab\div(a+b)$. **41.** $ab\div(a+b)$. **42.** $2(a+b)$.
43. $(a+b+c+d)\div(m+n)$. **44.** $(c^2-ab)\div(a+b-2c)$.
45. $\frac{1}{2}(a+b)$. **46.** $\frac{1}{3}(a+b+c)$. **47.** $\frac{1}{3}(a+b+c)$.
48. $ab+ac+bc$. **49.** $\frac{1}{3}(a+b+c)$.
50 $\{cd(a+b)-ab(c+d)\}\div(ab-cd)$

EXERCISE 56

1. -8 **2** 1 **3.** 3. **4.** 7. **5.** 5. **6.** 10. **7.** 5.
8. 7. **9.** 3 **10.** 5. **11.** 5. **12.** 5. **13.** 1.
14. -2. **15.** 36. **16.** 13. **17.** $\frac{80}{87}$. **18.** 7. **19.** 13.
20. 3. **21.** $1\frac{2}{7}$. **22.** 8. **23.** 4. **24.** $-3\frac{1}{2}$. **25.** 5.
26. $4\frac{1}{2}$. **27.** $a^2(b-a)\div b(a+b)$. **28.** $4\frac{1}{6}$. **29.** $4\frac{1}{3}$.
30. 3. **31.** $\frac{2}{5}$. **32.** $2\frac{21}{29}$. **33.** $2\frac{3}{4}$.
34. $(ab+bc-2ac)\div(a+c-2b)$. **35.** $\frac{2}{7}$. **36.** 7. **37.** 2.
38. $2(c-b-a)$. **39.** $6ab+3ac+2bc$. **40.** $ab+a+b$.
41. $ab+ac+bc$ **42.** $a^2+b^2+c^2$. **43** $-(a+b+c+d)$.
44. $-3a$. **45.** 10. **46.** $3a$.
47. $\{c^2(a-b)+b^2(a-c)\}\div(ab+ac-2bc)$.
48. $2ab\div(a+b)$. **49.** $1\frac{21}{29}$.
50. $\{b^2(c-a)+a^2(c-b)\}\div(ac+bc-2ab)$. **51.** 5.
52. 13. **53.** $\frac{1}{2}(a+b)$ **54.** $a^2\div(b-a)$.
55. $(bn-am)\div(m-n)$.

EXERCISE 57.

1. 4. **2.** 0. **3.** $1\frac{3}{4}$. **4.** 5. **5.** 10. **6.** 0.
7. 5. **8.** -9. **9.** 0. **10.** 3. **11.** 3. **12.** 0.
13. $1\frac{13}{17}$. **14.** -2. **15.** 5. **16.** 1. **17.** -10.

18. $\frac{1}{3}$. **19.** 4. **20.** $-b$. **21.** $2ab \div (a-b)$. **22.** 5.
23. 2. **24.** 4. **25.** 6. **26.** 1. **27.** $-9\frac{1}{2}$. **28.** 0.
29. $2b$. **30.** $\{ab(c+d)-cd(a+b)\} \div (ab-cd)$.
31. $(c^2-ab) \div (a+b-2c)$. **32.** $\frac{1}{2}(a+b-3)$. **33.** 4.
34. $1\frac{1}{2}$. **35.** $\frac{1}{2}(a+b)$. **36.** $-2\frac{1}{2}$. **37.** $2\frac{1}{2}$. **38.** 7.
39. $2(a+b)$. **40.** 8. **41.** $3\frac{1}{3}$. **42.** $\frac{5}{4}$. **43.** $\frac{55}{36}$.
44. $2\frac{1}{2}$. **45.** $\frac{5a}{2}$. **46.** $2\frac{3}{4}$. **47.** $8\frac{1}{2}$. **48.** 1.
49. $3a$. **50.** $(3ab-a^2-b^2) \div (a+b)$.

EXERCISE 58.

1. $\frac{1}{a}$. **2.** $\frac{(a+b)(n-m)}{2n-2m+a-b}$. **3.** $\frac{b+c+q+r}{a+p}$.
4. $\frac{cn}{m(b-a)}$. **5.** $\frac{cq-br}{ar-cp}$. **6.** $\frac{a(ab-cd)}{ad-bc}$. **7.** $\frac{3}{4}$.
8. $2(a+b)$. **9.** $\pm ac$. **10.** $\frac{1}{a}$. **11.** $-\frac{7}{8}$.
12. $ab \div (a+b)$. **13.** $-\frac{11}{3}$. **14.** $-6\frac{1}{2}$. **15.** -10.
16. $(-a^2+b^2+3ab) \div (a+b)$. **17.** 6. **18.** $2\frac{1}{2}$.

EXERCISE 59.

1. 4. **2.** 50. **3.** 25. **4.** $\frac{13}{81}$. **5.** $(a-b)^2$. **6.** a.
7. $6a$. **8.** 7. **9.** $\frac{17}{10}a$. **10.** 20. **11.** $(a-b)^2 \div (2a-b)$.
12. $\frac{a}{4}$. **13.** $\frac{4a}{5}$. **14.** $(a-b)^2 \div 2b$.
15. $a^2 - \frac{(b-2a)^3}{27b}$. **16.** 81. **17.** $\frac{a}{3}$. **18.** 25. **19.** 5.
20. $\frac{1}{a}\left(b-\frac{cn}{n-1}\right)^2$. **21.** 6. **22.** ± 1. **23.** 0.
24. 0. **25.** $2r-p^2-q^2$. **26.** $2c-a-b$. **27.** $10\frac{1}{2}$.
28. 5. **29.** 4. **30.** 1. **31.** 0. **32.** $\frac{1}{2}$. **33.** 15.
34. $\frac{1}{2}(b-c)$. **35.** $-\frac{\sqrt{a}}{2}$. **36.** $\frac{-\sqrt{a}}{1+a}$. **37.** $\frac{3}{1}$.
38. $\sqrt{\frac{2a-b}{a^2b}}$. **39.** $\frac{a(a^2+1)}{(a+1)^2}$. **40.** $\frac{4}{81}$. **41.** $\frac{4a}{3}$.
42. 1. **43.** $a(m^2+n^2) \div (m^2-n^2)$. **44.** $a(b-1)^2 \div 4b$.

45. $\frac{a}{2b}(b-1)^2$. **46.** $\frac{\sqrt[3]{p+1}+\sqrt[3]{p-1}}{\sqrt[3]{p+1}-\sqrt[3]{p-1}}$.

47. $\frac{\sqrt[4]{a+1}+\sqrt[4]{a-1}}{\sqrt[4]{a+1}-\sqrt[4]{a-1}}$. **48.** $\frac{b(p^2+1)^2+4p^2}{a(p^2-1)^2}$.

49. $\pm\frac{3a-1}{\sqrt{(1-a)(9a-1)}}$. **50.** 5. **51.** $\frac{1}{15}$. **52.** $2a^3$.

53. $\frac{3}{4}a$. **54.** 0. **55.** $(\sqrt[3]{b}-\sqrt[3]{a})^3$. **56.** $(b^{\frac{1}{3}}+b^{\frac{2}{3}})^3$.

57. $\left(\frac{4b^{\frac{1}{3}}-3a^{\frac{1}{3}}}{2}\right)^3$ **58.** 0. **59.** $\frac{4}{b-2}$. **60.** $-\frac{6}{7}$.

61. $\frac{27abc-(a+b+c)^3}{3(a^2+b^2+c^2-ab-bc-ac)}$. **62.** $\frac{a}{2}$. **63.** $-a$.

64. $\frac{9b}{2(\sqrt{5}-\sqrt{3})^3}$. **65.** ab. **66.** 5. **67.** $\frac{1}{p+1}$.

68. 1. **69.** $\left(\frac{p+q}{p-q}\right)^{22}$ or $\left(\frac{p-q}{p+q}\right)^{12}$. **70.** 78.

EXERCISE 60.

1. 10. **2.** 16 ; 2. **3.** 70. **4.** $H=$ Rs. 600 ; $G=$ Rs. 250.
5. 20 days. **6.** £1[illegible]. **7.** $1\frac{5}{13}$ hrs.
8. £10 2*s.* ; £5 1*s.* ; £4 16*s.*
9. (1) $38\frac{7}{11}m$. (2) $21\frac{9}{11}m$. (3) $5\frac{5}{11}m$ past 7.
10. 35 ; 25. **11.** 156 days. **12.** 240. **13.** 25 lbs.
14. 16 ; 20 coins. **15.** $\frac{10}{13}$. **16.** 48 of each.
17. $\frac{a+nq}{p+q}$; $\frac{pn-a}{p+q}$. **18.** 6 miles. **19.** 26.
20. 180 and 90 ; 230 and 100 yds. **21.** 15 ft. ; 12 ft.
22. 3 . **23.** 23. **24.** 14 miles. **25.** 7*s.* ; 11*s.* 8*d.*
26. 193. **27.** 14 miles from *B*. After 6 hrs.
28. $\frac{An}{(n+1)^2}-n$; $\frac{An}{(n+1)^2}+n$; $\frac{A}{(n+1)^2}$; $\frac{An^2}{(n+1)^2}$;
18, 22, 10 and 40.
29. The hound 960 leaps and the hare 1,200 leaps.
30. 180,000 men. **31.** 654. **32.** 2 hrs. after *B*.

33. 121 yds. **34.** 99 yds.; 77 yds. **35.** 1,000 men.
36. 760 men. **37.** 740; 1053 votes.
38. 10 hrs. 95 m. from L. **39.** 36 miles.
40. 2 miles an hour. **41.** 60 miles; Passenger train 30 miles an hour; Goods' train 20 miles an hour.
42. 150 miles; 40 miles an hour. **43.** 120 bushels.
44. 5 : 1. **45.** 600 men. **46.** 40. **47.** 4 hrs. and 6 hrs.
48. $6p$ and $4p$.

EXERCISE 61.

1. $x=5$; $y=3$. **2.** $x=1$; $y=5$. **3.** $x=5$; $y=4$.
4. $x=3$; $y=2$. **5.** $x=15$; $y=16$. **6.** $x=-2$; $y=\frac{1}{2}$.
7. $x=2$; $y=3$. **8.** $x=1$; $y=7$.

9. $x=\dfrac{cq-br}{aq-bp}$; $y=\dfrac{ar-cp}{aq-bp}$.

10. $x=\dfrac{ac+bd}{a^2+b^2}$; $y=\dfrac{ad-bc}{a^2+b^2}$.

11. $x=\dfrac{bc}{a+b}$; $y=\dfrac{ac}{a+b}$. **12.** $x=\dfrac{bn+dm}{ad+bc}$; $y=\dfrac{an-cm}{ad+bc}$.

13. $x=\frac{1}{5}$; $y=\frac{1}{2}$. **14.** $x=\dfrac{ac(dn+bm)}{ad+bc}$; $y=\dfrac{bd(cn-am)}{ad+bc}$.

15. $x=\frac{1}{4}$; $y=\frac{3}{5}$. **16.** $x=\frac{8}{1705}$; $y=\frac{303}{682}$. **17.** $x=7$; $y=1$.

18. $x=\dfrac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$; $y=\dfrac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

19. $x=21$; $y=20$. **20.** $x=2$; $y=3$. **21.** $x=3$; $y=5$.

22. $x=\dfrac{c-d-a+b}{2(bc-ad)}$; $y=\dfrac{a+b-c-d}{2(bc-ad)}$.

23. $x=\dfrac{c(a^2+b^2)}{a^2-b^2}$; $y=\dfrac{c(a+b)}{2a}$.

24. $x=\dfrac{b^2-c^2}{a(a-c)}$; $y=\dfrac{b^2-ac}{b(c-a)}$. **25.** $x=\dfrac{n}{a^2-b^2}$; $y=\dfrac{n}{b^2-a^2}$.

26. $x=\dfrac{ab}{a+b}$; $y=\dfrac{ab}{b-a}$. **27.** $x=\dfrac{b-a}{b+a}$; $y=\dfrac{a(a+b)^2-4b^3}{a(a^2-b^2)}$.

28. $x=y=1$. **29.** $x=a+b$; $y=a-b$. **30.** $x=a$; $y=b$.
31. $x=y=1$. **32.** $x=8$; $y=4$. **33.** $x=4$; $y=2$

34. $x=\dfrac{b^2+c^2-a^2}{2a}$; $y=\dfrac{c^2+a^2-b^2}{2b}$ **35.** $x=\frac{1}{2}$; $y=\frac{1}{3}$.

36. $x=b-c$; $y=c-a$. **37.** $x=\dfrac{b-c}{a-b}$; $y=\dfrac{c-a}{a-b}$.

38. $x=1$; $y=-1$. **39.** $x=1=y$. **40.** $x=a^2$; $y=b^2$.

41. $x=3$; $y=2$. **42.** $x=\frac{4}{5}a$; $y=\frac{5}{4}a$.

43. $x=-\frac{16}{15}a$; $y=\frac{7}{15}a$. **44.** $x=(a+b)^2$; $y=(a-b)^2$.

45. $x=y=343$. **46.** $x=a+b$; $y=a-b$. **47.** $x=a$; $y=b$.

48. $x=\dfrac{b^2(a^2+1)(a^4-1)}{(a^2b^2-1)(a^4-b^4)}$; $y=\dfrac{a^2(b^2+1)(b^4-1)}{(a^2b^2-1)(a^4-b^4)}$.

49. $x=\dfrac{a(a^2+ab+b^2)}{b(a+b)}$; $y=-\dfrac{a^2}{a+b}$. **50.** $x=-ab$; $y=a+b$.

EXERCISE 62.

1. 1; 2; 3. **2.** 1; 2; 3. **3.** 35; 30; 25.

4. 2; 3; 4. **5.** 7; 10; 9. **6.** 24; 60; 120.

7. 1; 2; 3. **8.** 8; 10; 12. **9.** 5; 7; -3.

10. 24; 60; 120. **11.** $x=y=z=1$. **12.** $x=\dfrac{2mnpa}{n(p+m)-pm}$, &c.

13. $\frac{1}{7}$; $\frac{1}{63}$; $\frac{1}{77}$. **14.** $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$. **15.** $1\frac{1}{6}$; $3\frac{1}{2}$; $2\frac{1}{10}$.

16. 10; 6; 2. **17.** -2; $-\frac{7}{3}$; $\frac{1}{3}$.

18. $\dfrac{1}{(c-a)(c-b)}$; $\dfrac{1}{(b-a)(b-c)}$; $\dfrac{1}{(a-b)(a-c)}$.

19. $\dfrac{1}{x}=\dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right)$; $\dfrac{1}{y}=\dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{c}-\dfrac{1}{b}\right)$; $\dfrac{1}{z}=\dfrac{1}{2}\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)$.

20. abc; $ab+ac+bc$; $a+b+c$. **21.** $x=\dfrac{b^2+c^2-a^2}{2bc}$, &c.

22. $x=y=z=\dfrac{a+b}{c}$. **23.** $\dfrac{6-b}{2b-3}$; $\dfrac{3}{2b-3}$; $\dfrac{6-2b}{2b-3}$.

24. $\dfrac{1}{x}=-\dfrac{1}{2}\left(\dfrac{1}{b}+\dfrac{1}{c}\right)$, &c. **25.** $x=y=z=1$.

26. $x=\dfrac{b^2+c^2-a^2}{2bc}$, &c. **27.** 1; 2; 3; 4.

28. $\frac{1}{6}$; $-\frac{1}{6}$; $\frac{1}{18}$; $-\frac{1}{18}$. **29.** 1; 2; 3; 4. **30.** 1; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$.

EXERCISE 63.

1. $x=3$; $y=4$. 2. $x=12$; $y=5$. 3. $x=12$; $y=13$.
4. $x=14$; $y=20$. 5. $x=3$; $y=2$. 6. $x=\frac{q+ap}{a^2+1}$; $y=\frac{p-aq}{a^2+1}$.
7. $x=2$; $y=-3$; $z=1$. 8. $x=1$; $y=3$; $z=4$.
9. $x=c-b$; $y=a-c$; $z=b-a$. 10. $x=b-c$; $y=c-a$; $z=a-b$. 11. $x=\frac{a(b+c)}{(a-b)(c-a)}$, &c. 12. $x=b+c-a$; $y=c+a-b$; $z=a+b-c$. 13. $x=\frac{1}{a}$; $y=\frac{1}{b}$; $z=\frac{1}{c}$
14. $x=a$; $y=b$; $z=c$. 15. $x=\frac{1}{(a-b)(a-c)}$, &c
16. $x=b+c-a$; $y=c+a-b$; $z=a+b-c$. 17. $x=\frac{1}{a}$; $y=\frac{1}{b}$; $z=\frac{1}{c}$. 18. $x=a$; $y=b$; $z=c$ 19. $x=a$; $y=b$; $z=c$.
20. $x=a^2-b^2$; $y=c^2-a^2$; $z=b^2-c^2$.

EXERCISE 64

1. $x=1-b$; $y=b$ 2. $x=\frac{1}{\sqrt{2}}$; $y=\sqrt{2}$.
3. $x=4$; $y=3$. 4 $x=5b$; $y=4b$. 5. $x=y=2$.
6. $x=2$; $y=0$. 7. $x=7$; $y=5$. 8. $x=10$; $y=1$.
9. $x=\frac{2}{a}$; $y=\frac{2}{b}$; $z=\frac{2}{c}$. 10. $x=1$; $y=2$; $z=3$.
11. $x=1$; $y=2$; $z=3$. 12. $x=p$; $y=q$; $z=r$.
13. $z=\sqrt{\frac{(c+a-b)(a+b-c)}{2(b+c-a)}}$, &c. 14. $x=y=z=a$.
15. $x=2$; $y=3$; $z=4$. 16. Consider $\frac{1}{xy}$, $\frac{1}{yz}$, and $\frac{1}{zx}$ as the unknown quantities. 17. $x=\frac{1}{2}\left(\frac{ac}{b}+\frac{ab}{c}-\frac{bc}{a}\right)$.
18. $\frac{1}{x^2}=\frac{1}{2}(b^2+c^2-a^2)$.

EXERCISE 65.

1. 24 ; 6. **2.** A's 58 ; B's 18. **3.** $\frac{1}{18}$. **4.** 82. **5.** 376.
6. 2*s.* 4*d.* ; 3*d.* **7.** $106\frac{2}{33}$ yds. **8.** 457.
9. 36 and 27 miles per hour; 756 miles. **10.** 27*s.* ; 15 beggars.
11. £2,700 ; 9 men. **12.** £56 ; £33. **13.** $4\frac{7}{8}$ *m.* **14.** $\frac{5}{20}$; $\frac{9}{16}$.
15. A in $\frac{2abc}{bc+ac-ab}$ days ; B in $\frac{2abc}{ab+bc-ac}$ days ; C in $\frac{2abc}{ca+ab-bc}$. **16.** 4 yds. ; 5 yds. **17.** 72 lbs. **18.** $40\frac{5}{22}$ *m* ; $34\frac{17}{22}$ *m.* **19.** $17\frac{5}{36}$; $12\frac{5}{36}$; $14\frac{17}{36}$ gals. **20.** $6m$; $5\frac{5}{11}$ *m.*
21. 150 *m.* **22.** $3m$. **23.** 33 *m* ; 6 *m.* ; $5m$. **24.** A. $1\frac{7}{8}$ *m* ; B, $1\frac{1}{8}$ *m.* **25.** 27 *m* ; $3\frac{1}{2}$ *m.* **26.** 12 ft. ; 9 ft. **27.** 255.
28. £6 ; 20 persons. **29.** 8 and 12 gals. **30.** 90 miles.
31. A, in 5 minutes ; B, in 6 minutes.
32. $2\frac{1}{2}$ and 2 miles ; distance 5 miles.
33. 100 *m* and 25 *m.* **34.** $b(n-1)\div(a-c)$ miles.
35. 50 lbs.: £70. **36.** 6 and 10 qrs. **37.** 45 *m* and $22\frac{1}{2}$ *m.*
38. 10 and 12 *m* an hour. **39.** 12 *m.* **40.** $33\frac{1}{3}$ *m* ; $48\frac{1}{3}$ *m.*
41. 45 and 30 miles. **42.** 30 and 50 miles.
43. 4550 I; 1750 C ; 3853 A. **44.** 432.
45. Rs. 1,250 ; 1,500 ; 1,250. **46.** 63 ; (20 ; 21 ; 22).
47. 36 *m* ; 12 *m.*

EXERCISE 66.

1. ± 1. **2.** $\pm\sqrt{5}$. **3.** $\pm\frac{1}{\sqrt{5}}$. **4.** $\pm\frac{a}{\sqrt{2}}$ **5.** $\pm\frac{1}{2}$.
6. $\pm 2\sqrt{\frac{3}{5}}$. **7.** ± 1. **8.** $\frac{4}{3}$; 0. **9.** ± 3. **10.** ± 2. **11.** ± 6.
12. ± 1. **13.** $\pm\frac{a(n-1)}{\sqrt{2n-1}}$. **14.** 4 ; 0. **15.** $-2a$. **16.** $\pm\frac{5}{6}$.
17. $\pm\sqrt{2ab-b^2}$. **18.** $\pm\frac{5}{8}$. **19.** $\pm\frac{a(b-c)}{2\sqrt{bc}}$.
20. $\pm\sqrt{\frac{a^2-pb^2}{p+1}}$ where $p=\left(\frac{c+d}{c-d}\right)^2$. **21.** $\pm\frac{a\sqrt{3}}{2}$.
22. $\pm\sqrt{3}$. **23.** $\pm\frac{3}{5\sqrt{19}}$. **24.** $\pm a$. **25.** $\pm\frac{4}{\sqrt{3}}$.

EXERCISE 67.

1. $4;\ -2.$ **2.** $20;\ -6.$ **3.** $8;\ -40.$ **4.** $1;\ -8.$
5. $1;\ -20.$ **6.** $25;\ -136.$ **7.** $10;\ 2.$ **8.** $3;\ -1.$
9. $7;\ -\frac{3}{2}.$ **10.** $\frac{5}{3};\ -\frac{3}{2}.$ **11.** $2;\ \frac{1}{2}.$ **12.** $-18\pm 6\sqrt{3}.$
13. $3;\ -\frac{4}{5}.$ **14.** $2;\ \frac{12}{25}.$ **15.** $\frac{1}{8}(27\pm\sqrt{57}).$ **16.** $11;\ -13.$
17. $1;\ 10\frac{2}{3}.$ **18.** $7;\ -1\frac{3}{7}.$ **19.** $5\pm\sqrt{30}.$
20. $a^2x^2-x(a^2b-ab^2+bcd+acd)-bc(ab-cd-ad)=0.$

21. $b;\ \dfrac{a^2}{b}$ **22.** $a;\ \dfrac{1}{a}.$ **23.** $-(a+b);\ b-a.$

24. $-a;\ -3ab.$ **25.** $a;\ b.$ **26.** $0;\ \dfrac{2ab}{a+b}.$

27. $c;\ -\dfrac{a^2+ac+b^2+bc}{a+b+2c}.$ **28.** $\dfrac{a}{2}(-3\pm\sqrt{3}).$

29. $2a-b;\ 3b-a.$ **30.** $\dfrac{a+b}{2}\pm\dfrac{a-b}{2}\times\sqrt{\dfrac{p-2q}{p+2q}}.$

31. $2a-b;\ 2b-a.$ **32.** $\pm\sqrt{(a^2+10ab+b^2)}.$

33. $\pm\sqrt{(a^2+b^2)}.$ **34.** $\dfrac{a^2+b^2}{2b};\ \dfrac{2ab+b^2-a^2}{2a}.$

35. $0;\ \dfrac{2ab-bc-ac}{a+b-2c}$ **36.** $a+b;\ \dfrac{2ab}{a+b}.$ **37.** $0.$

38. $\dfrac{a}{b};\ \dfrac{b}{a}$ **39.** $\dfrac{1\pm\sqrt{(1+4ac)}}{2(a+b)}$ **40.** $5\pm\sqrt{19}.$

41. $\dfrac{a}{6}(-11\pm\sqrt{13}).$ **42.** $-a;\ -b.$ **43.** $\dfrac{a+b}{a-b};\ \dfrac{a-b}{a+b}.$

44. $\dfrac{1}{2ab}\{a^2+b^2\pm\sqrt{(a^2-b^2)^2+4abc^2}\}.$ **45.** $\dfrac{1}{a+b+c}$
$\times\{ab+ac+bc\pm\sqrt{(a^2b^2+b^2c^2+c^2a^2-abc(a+b+c))}\}.$
46. $49;\ 9.$ **47.** $18;\ 3.$ **48.** $4;\ -\frac{44}{5}.$ **49.** $\frac{1}{4};\ \frac{17}{4}.$
50. $-(2a-3b)\pm\sqrt{(5b^2-12ab-12b^2)}.$ **51.** $1;\ -8\frac{1}{3}.$

52. $\sqrt{3}+1;\ -\frac{1}{2}(\sqrt{3}+1).$ **53.** $\dfrac{2+\sqrt{3}}{2}(1\pm\sqrt{-3}).$

54. $\dfrac{1}{5-\sqrt{2}};\ -\dfrac{2}{5-\sqrt{2}}.$ **55.** $8;\ 20.$ **56.** $5;\ -8.$

57. $7;\ -1.$ **58.** $11;\ -5.$ **59.** $3;\ -\frac{1}{17}.$ **60.** $2\frac{1}{3};\ -\frac{13}{7}.$

61. $\frac{1}{2}(2a+b+b\sqrt{1+4a^2})$ **62.** $3;\ 4.$ **63.** $0;\ 1\frac{4}{5}.$ **64.** $1.$

65. $0;\ -2\frac{1}{2}$. 66. b.

67. Multiply out and cancel terms involving x^2.

68. $1;\ 3$. 69. $\frac{1}{2}(-1\pm\sqrt{33})$.

70. $\frac{1}{2}(-1\pm\sqrt{-3}),\ \dfrac{-(1-b)\pm\sqrt{(1-b)(1+7b)}}{2(1-b)}$. 71. $5;\ 3$.

72. $1;\ \sqrt{2}+\sqrt{3}+2$.

73. $\frac{1}{2}\left\{\dfrac{m^2+n^2}{m^2-n^2}\pm\sqrt{\left(\dfrac{m^2+n^2}{m^2-n^2}\right)^2-\dfrac{4mn}{m^2-n^2}}\right\}$.

74. $(m+1)x^2-(m^2-2m+5)x-(m^2-m-2)=0$.

75. $0;\ \dfrac{1}{a}+\dfrac{1}{b}$. 76. $a;\ \dfrac{a}{8}(-3\pm\sqrt{-7})$. 77. $0;\ \pm\dfrac{\sqrt{2a-1}}{a}$.

78. $\dfrac{2(a^2+b^2)+2(a+b)\sqrt{a^2-ab+b^2}}{(a-b)^2}$ 79. $0;\ -2;\ \frac{1}{3}$.

80. $\pm 2;\ \pm\sqrt{-2}$.

EXERCISE 68.

1. $5;\ 1$. 2. $-\frac{7}{3},\ 1$. 3. $0;\ \frac{3}{4}$. 4. $-\frac{7}{6},\ 0$.

5. $1-a;\ a^2-a+1$. 6. $\dfrac{b^2-c^2}{b^2-a^2};\ \dfrac{c^2-a^2}{a^2-b^2}$.

7. $x^2-x-12=0$. 8. $x^2+(a-b)x-ab=0$.

9. $x^2-2x+1-a^2=0$. 10. $x^2+(c-a)x+(a-b)(b-c)=0$.

11. $x^2+(a+c)bx+ab^2c=0$. 12. $x^2-4x+1=0$.

13. $-\dfrac{b}{c}$. 14. $\dfrac{b^2}{ac}-2$. 15. $\dfrac{b^2}{c^2}-\dfrac{2a}{c}$. 16. $\dfrac{3b}{a}-\dfrac{b^3}{a^2c}$.

17. $\dfrac{b^2-3ac}{a^2}$. 18. $\dfrac{b^2-ac}{a^2}$. 19. $\dfrac{(b^2-ac)(b^2-3ac)}{a^4}$.

20. $(x-1)(a^2x-b^2x+a^2-c^2)$. 21. $(x+a-b)(x+b-c)$.

22. $(x-1)\{(b+c-2a)x-a-b+2c\}$.

23. $(x-1)\{ax(b-c)-c(a-b)\}$.

24. $(x-4-\sqrt{5})(x-4+\sqrt{5})$. 25. $(x+a^2+ab)(x+b^2+ab)$.

26. $(x+a^2-ab)(x+b^2-ab)$.

EXERCISE 69.

1. $2;\ \sqrt[3]{-6}$. 2. $(x^2-x)^2-(x^2-x)-132=0$.

3. $\pm 3;\ \pm\frac{1}{3}\sqrt{-\frac{11}{3}}$. 4. $x^n=\frac{1}{2}(m\pm\sqrt{m^2+4p})$. 5. $81;\ 9$.

6. 1 ; 4. **7.** 0 ; $\frac{63}{65}m$. **8.** 0 ; $\dfrac{2^m-1}{2^m+1}$. **9.** -1 ; $\dfrac{1\pm\sqrt{-3}}{2}$.

10. $x+\dfrac{1}{x}=\dfrac{-2a\pm\sqrt{2a+2}}{a-1}$. **11.** $x^2-\dfrac{1}{x^2}=\dfrac{-2\pm\sqrt{1-a^4}}{a^2}$.

12. 2. **13.** $1\frac{1}{3}$. **14.** 1. **15.** $\frac{1}{2}$.

16. $y^2+3y-4=0(y=\sqrt{x^2+5x+7})$.

17. $y^2+y-20=0(y=\sqrt{x^2+9x+3})$.

18. $y^2+y-6=0(y=\sqrt{x^2-5})$.

19. $y^2-2y-35=0(y=\sqrt{x^2-8x+40})$.

20. $y^2+6y-16=0(y=\sqrt{2x^2-3x+2})$.

21. $y^2-\frac{1}{2}y-1\frac{1}{2}=0(y=\sqrt{x^2-2x-3})$.

22. 9 ; -2 ; $\frac{1}{2}(7\pm\sqrt{173})$. **23.** 1. **24.** 4 ; -9.

25. 1 ; -4 ; $\frac{1}{2}(-3\pm\sqrt{109})$. **26.** 2 ; -5 ; $\frac{1}{2}(-3\pm\sqrt{241})$.

27. 0 ; 2. **28.** 1 ; $\frac{1}{2}$. **29.** 2 ; $-\frac{1}{2}$.

30. $y^2-2y+1=0$ where $y=\sqrt{3x^2-2ax+4}$.

31. $\left(\dfrac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$; $\left(\dfrac{a-b}{a+b}\right)^{\frac{2pq}{q-p}}$ **32.** 3, 2. 33. 2 ; -1.

34. ± 1. 35. $\dfrac{w^2}{1-w^2}$; $\dfrac{w^2}{1-w}$; w = imaginary cube root of 1.

36. 2 ; 6 ; $3\pm\sqrt{21}$. **37.** Divide by x^2 ; and put $x-\dfrac{1}{x}=y$.

38. ± 1 ; 2 ; $-\frac{1}{2}$. **39.** $\frac{3}{2}(1\pm\sqrt{5})$; $3(1\pm\sqrt{2})$.

40. 1 ; $\frac{1}{2}(-3\pm\sqrt{5})$. **41.** 1 ; 1 ; $\pm\sqrt{-1}$.

42. $\frac{1}{6}(7\pm\sqrt{85})$; $\frac{1}{3}(1\pm\sqrt{10})$. **43.** $x+\dfrac{1}{x}=\frac{10}{3}$ or $-\frac{16}{3}$.

44. Divide by x^2 ; and put $x+\dfrac{1}{x}=y$. **45.** 1 ; 4 ; $\pm\sqrt{-1}$.

46. 1 ; 2 ; $-\frac{1}{2}(5\pm\sqrt{17})$. **47.** 4 ; 2 ; 1 ; $\frac{1}{2}$.

48. 6 ; -1 ; $\frac{1}{2}(5\pm\sqrt{-39})$. **49.** $8a$; $-a$; $\dfrac{a}{2}(7\pm\sqrt{-71})$.

50. $\frac{3}{2}$; $\frac{1}{2}(3\pm\sqrt{10})$.

51. $\frac{a}{2}(5\pm\sqrt{13})$; $\frac{a}{2}(5\pm\sqrt{-3})$.

52. $x^2+5ax=-5a^2\pm\sqrt{a^4+c^4}$.

53. $\left(x+\frac{c}{ax}\right)^2+a\left(x+\frac{c}{ax}\right)+b=\frac{2c}{a}$.

54. $(x^2-4ax)^2-2a^2(x^2-4ax)-15a^4-16c^4=0$.

55. $(x^2-9x)^2+38(x^2-9x)+336=0$.

56. $(x^2+21x)^2+158(x^2+21x)+5565=0$.

57. $x^2+(2a+3b)x+a^2+3ab+b^2=\pm\sqrt{(b^4+c^4)}$.

58. $y^2-y+7\frac{1}{2}=0$; $(y=\sqrt{x^2+4x-5})$. **59.** -4.

60. 1; $1+\frac{1}{128}(3\pm\sqrt{-7})^4$. **61.** $x^2+3x=\frac{1}{4}$ or $-\frac{9}{4}$.

62. $x=5$ (obvious). **63.** $x=5$ (obvious). **64.** 8; $-\frac{23}{5}$.

65. ± 5. **66.** 1, $\frac{47-44\sqrt{6}}{23}$. **67.** 1; $\frac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}$.

68. $(x^2+5x)^2+6(x^2+5x)-40=0$.

69. $x^2+3x=4$ or -2. **70.** $4x^2-3x=1$ or $2\frac{1}{2}$.

71. $x^2-5x+7=0$ or 1. **72.** $x^2+x=6$ or -7.

73. $\frac{2}{3}(-4\pm\sqrt{34})$. **74.** $\frac{1}{2}(-3\pm\sqrt{21})$. **75.** 2.

76. $x^2+x=\frac{1}{2}(-1\pm\sqrt{3})$. **77.** 4, 3; $\frac{1}{4}$; $\frac{1}{3}$.

78. 1; $\frac{-47\pm\sqrt{-1655}}{46}$. **79.** $x^2+5x=36$ or 1.

80. $\frac{3}{2}\pm\sqrt{\left(\frac{25\pm\sqrt{1345}}{72}\right)}$. **81.** $\frac{1}{2}$; $-4\frac{1}{2}$; $-2\pm\sqrt{-6}$.

82. 1; -3; $-\frac{1}{2}$. **83.** $\left(\frac{p+q}{p-q}\right)^{12}$; $\left(\frac{p-q}{p+q}\right)^{12}$. **84.** $\pm m\frac{n^5+1}{n^5-1}$.

85. 0; -3; 12; -15. **86.** 4; $3\sqrt[3]{3}$. **87.** 4; -1; $\frac{4}{3}$; $\frac{5}{3}$.

88. $\frac{2}{1-\sqrt[3]{2}}$. **89.** m. **90.** a, b, c; $(x+a)(x+b)(x+c)=x^3$.

91. $-(a+b)$; $\frac{1}{a+b}\times\{-(a^2+b^2)\pm\sqrt{(a+b)^4+4a^2b^2}\}$.

92. 2. **93.** 2; $2\pm\sqrt{3}$. **94.** 0; $\frac{4}{3}a$. **95.** 1; -3.

96. $x^2+3x=\frac{2}{9}$ or $-\frac{20}{9}$. **97.** 0; $\frac{2}{3}$; $\frac{1}{3}(1\pm\frac{1}{2}\sqrt{-3})$.

98. 1; $(x-1)^2=\frac{1}{2}(1\pm\sqrt{65})$. **99.** 1; $2\pm\sqrt{2}$; $2\pm\sqrt{-1}$.

100. 0; $\pm\sqrt{-5}$; $\pm\sqrt{\frac{11}{3}}$.

EXERCISE 70.

1. $x=3\frac{5}{7}$; $y=10\frac{74}{169}$. 2. $x=\pm 3$; $y=\pm 1$. 3. $x=3$; $y=1$.
4. $3x^2y^2-196xy+2380=0$. 5. $x=\pm 7$; $y=\mp 2$.
6. $x=1$ or $1-(1\mp\sqrt{2})^3$; $y=0$ or $-1\pm\sqrt{2}$.
7. $x=5$ or $-2\frac{1}{2}$; $y=8$ or $\frac{1}{2}$.
8. $xy=4$; $x-y=-9$. 9. $x=\pm 6$; $y=\pm 3$.
10. $x=-\frac{52}{17}a$; $y=-\frac{13}{17}a$. 11. $x=\pm 7$; $y=\pm 3$.
12. $x=4$; $y=2$. 13. $x=3$ or $-\frac{1}{3}$; $y=3$ or $-\frac{1}{3}$.
14. $x=1$ or $\frac{5}{3}$; $y=2$ or $\frac{2}{3}$. 15. $x=-1$ or 3; $y=1$ or -3.
16 $x=\pm\dfrac{a(m+n)}{\sqrt{m^2+n^2}}$; $y=\pm\dfrac{a(m-n)}{\sqrt{m^2+n^2}}$.
17. $xy=\dfrac{b^2-a^2}{2}$; $x+y=b$. 18. $x=3$; $y=2$.
19. $x=0$; 1; $\pm\dfrac{1}{\sqrt{2}}$; $y=0$; 2; $2\pm\sqrt{2}$.
20. $x=\pm 2$; $y=\pm 1$.
21. $xy=\frac{1}{10}\left(5a^2\pm\sqrt{5a^4+\dfrac{20b}{a}}\right)$ and $x+y=a$.
22. $x=\dfrac{m+n}{mn+1}$ or $\dfrac{mn+1}{m+n}$, $y=\dfrac{n-m}{mn-1}$ or $\dfrac{mn-1}{n-m}$.
23 $x=3a$ or $-2a$; $y=2b$ or $-3b$. 24. $x=2$; $y=1$.
25. $x=3$; -4; 2 or -5; $y=2$; -5; 3 or -4.
26. $x=2$; $y=1$. 27. $x=y=1$. 28. $x=7$: $y=11$.
29. $x=4$; $y=1$. 30. $x=8$; $y=2$.
31. $x=3$; $y=-3$. 32. $x=125$; $y=1$.
33. $x=2$; $y=1$ 34. $x=3$; $y=2$. 35. $x^3=\dfrac{a^9}{a^6+b^6}$; $y^3=\dfrac{b^9}{a^6+b^6}$. 36. $x=a$; $y=b$. 37. $x=4$; $y=2$.
38. $x=4$; $y=5$. 39. $x=\frac{a}{2}(1\pm\sqrt{3})$; $\frac{a}{2}\left(1\pm\dfrac{1}{\sqrt{3}}\right)$; $y=\frac{a}{2}(1\pm\sqrt{3})$; $\frac{a}{2}\left(1\pm\dfrac{1}{\sqrt{3}}\right)$. 40. $x=a$; b; $y=b$; a.

EXERCISE 71.

1. $x=1$; $y=1$; $z=2$.

2. $x=\pm\dfrac{a^2}{\sqrt{a^2+b^2+c^2}}$, $y=\pm\dfrac{b^2}{\sqrt{a^2+b^2+c^2}}$;

$$z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}.$$

3. $x=4$; $y=1$; $z=2$. 4 $x^2=\dfrac{a(b+c)}{(a-b)(c-a)}$; &c.

5. $x=b-c$; $y=c-a$; $z=a-b$. 6. $x=1$; $y=2$; $z=3$.

7. $x=2$; $y=3$; $z=4$. 8. $x=1$; $y=2$; $z=3$.

9. $x=\{b+c-a+\sqrt{\tfrac{1}{3}(2bc+2ac+2ab-a^2-b^2-c^2)}\}\div 2p$

where $2p^2=a+b+c+3\sqrt{\tfrac{1}{3}(2bc+2ac+2ab-a^2-b^2-c^2)}$.

10. $x=b+c$; or $b+c+\dfrac{2abc}{a-b-c}$, &c.

11. $x=b-c$; $y=c-a$; $z=a-b$.

12. $x=\pm\dfrac{a(b+c-a)}{\sqrt{(b+c-a)(c+a-b)(a+b-c)}}$, &c

13. $x=a^{\frac{1}{9}}b^{-\frac{5}{9}}c^{\frac{1}{3}}$; $y=a^{-\frac{7}{9}}b^{\frac{8}{9}}c^{-\frac{1}{3}}$, $z=a^{\frac{4}{9}}b^{\frac{8}{9}}c^{\frac{1}{3}}$.

14 $x=\pm\sqrt{\tfrac{1}{2}(a\pm\sqrt{a^2-(b-c)^2}}$; &c

15 $x=y=z=\pm\sqrt{\dfrac{c}{2}}$ 16 $x=a$, $y=b$, $z=c$.

17. $\dfrac{x^2}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$

$$=\frac{2(a^2+b^2+c^2)}{3(a^2+b^2+c^2)-2(bc+ca+ab)} \text{ or } 0.$$

18. $x=6$; $y=7$; $z=8$. 19. $x=3$; $y=1$, $z=2$.

20. $x=3$, $y=-2$; $z=0$.

EXERCISE 72.

1. 7; 3. 2. 20; 20. 3. 9; 8. 4. $\dfrac{1\pm\sqrt{5}}{2}$; $\dfrac{3\pm\sqrt{5}}{2}$.

5. $\dfrac{-1\pm\sqrt{-3}}{2}$. 6. 1; 6. 7. 44, 110. 8 24; 40.

9. 220; 165. **10.** 50. **11.** 16. **12.** 10; 15.
13. 20; 25. **14.** 2; 5; 8. **15.** 3. **16.** 400; 1600.
17. 3; 4. **18.** 252. **19.** 73 or 37. **20.** $\frac{3}{7}$. **21.** 5*m*.
22. 248. **23.** 54; 10. **24.** 2*d*.; 3*d*. **25.** $\frac{\sqrt{5}}{2}$; $\frac{\sqrt{5}(1+\sqrt{5})}{2}$.
26. 130. **27.** £40. **28.** 6; 9. **29.** 15. **30.** 25; 5.
31. 117; 130. **32.** $13\frac{3}{4}$ ft.; $16\frac{1}{2}$ ft. **33.** 10; 15. **34.** 30*m*.
35. *A* 4·30; *B* 5 A.M. **36.** 10. **37.** Bisect it.
38. −1; −3; 3. **39.** 66; 22. **40.** 9*d*. **41.** 28.
42. *A* 40 hrs.; *B* 60 hrs. **43.** 5 hrs; 3 hrs
44. 160 qrs.; £2. **45.** 24 days; *A* 4*s*.; *B* 3*s*. **46.** 10; 12.
47. 30 yds.; 40 yds. **48** 100. **49.** 1 : 9; 1 : 4.
50. £60; £40.

EXERCISE 74.

11. $(a+b+c)(a-b)(b-c)(c-a)$.
12. $(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+ac+bc)$.
13. $(a^2-b^2)(b^2-c^2)(a^2-c^2)$. **14.** $(a+b)(b+c)(c+a)$.
15. $3abc(a+b)(b+c)(c+a)$.
16. $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.
17. $(a-b)(b-c)(c-a)(ab+ac+bc)$.
18. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+ac+bc)$.
19. Same as *Q*. 17.
20. $(a-b)(b-c)(a-c)$
$\times(a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+abc)$.
21. $l=21$; $m=-76$; $n=60$.
22. $A=-8$; $B=-12$; $C=20$.

EXERCISE 75.

1. $a^3+ab+bc=0$. **2.** $\sqrt{\frac{q}{p}}=\sqrt[3]{\frac{s}{r}}$.
3. $(br-cq)(aq-pb)=(pc-ar)^2$. **4.** $p^2-q^2=1$
5. $\sqrt[3]{\frac{b}{a}}=\sqrt[4]{\frac{q}{p}}$. **6.** $la^2-am+n=9$, **7.** $p^2-q^2=4$.

8. $a^2-b^2=4pq$. **9.** $(bc_1-b_1c)(ab_1-a_1b)^2=(a_1c-ac_1)^3$.

10. $(qn-mr)^2(pm-lq)=(lr-pn)^3$. **11.** $\dfrac{A-B}{A-C}=\dfrac{a-b}{a-c}$.

12. $a(b^2-c^2)+2b(c^2-a^2)+4c(a^2-b^2)=0$.

13. $5(a^3-b^3)(2a^3+b^3)=9a(a^5-c^5)$.

14. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^3=c^{\frac{3}{2}}$. **15.** $(a-b)^2(a^2+b^2)=a^2b^2$.

16. $(a+b)^{\frac{2}{3}}-(a-b)^{\frac{2}{3}}=(8c)^{\frac{2}{3}}$. **17.** $b^2=ac$.

18. $a^4-4ac^3+3b^4=0$. **19.** $a-2a^2b^2-b^3+2c^4=0$.

20. $q^2+4rp=k^2(p+r)^2$. **21.** $ab+bc+ca+2abc=1$.

22. $\dfrac{2}{c}=\dfrac{1}{a}+\dfrac{1}{b}$. **23.** $2p^3-3pq^2-2r^3+2s^3=0$.

24. $d^2(a+b+c)+abc=0$

25. $a^2+b^2+c^2+2abc=1$. **26.** $a^2+b^2+c^2\pm 2abc=1$.

27. $a^3+b^3+c^3-3abc=0$.

28. $a^2b^2c^2(a^3+b^3+c^3+2abc)=a^3b^3c^3$.

29. $a^2+b^2+c^2=-2(ab+ac+bc)$. **30.** $abc=(4-a-b-c)^2$.

MISCELLANEOUS EXAMPLES.

5. $\dfrac{1}{p}$. **6.** $(a+nb)x^n$. **7.** $(a+b)(b+c)(c+a)(a+b+c)$.

9. (i) $r=\dfrac{abc}{(a-b)(a-c)}$; &c. (ii) $\pm(b+c)$; $\pm(b-c)$.

(iii) $x=2$ or -4; $y=3$ or -5; $z=4$ or -6.

12. $(x^2+2)(x^2+2x+2)$. **13.** $(n+1)(n+2)x^n$. **14.** $1-\sqrt{3}$.

17. $2a^{\frac{x+y}{2}}+a^{-\frac{x+y}{4}}+1$. **22.** 1. **23.** (i) $x=y=z=12$;

(ii) $\sqrt{x}=\sqrt{y}$; or $\sqrt{x}+\sqrt{y}=a$. **24.** 2 miles.

26. $(x-4)(x-3)(x-\frac{1}{4})(x-\frac{1}{3})$.

27. (i) $x=\dfrac{a-b}{1+m}$; $y=\dfrac{a}{1-m}$; $z=\dfrac{b}{1-m}$.

(ii) $[-(ab+ac+bc)\pm\sqrt{\{(ab+ac+bc)^2-4abc(a+b+c)\}}]$
$\div 2(a+b+c)$. **29.** $\dfrac{x}{(x-a)(x-b)(x-c)}$. **30.** 3; 2.

31. $1+5x+15x^2+45x^3+145x^4$.

32. (i) -4. (ii) $x^2+x=\frac{1}{2}(1\pm\sqrt{5})$.

35. $x(x-1)^2$; $(x^4+x^2-2)(x-3)$.

36. (i) $\dfrac{1-a^2-a^3}{a(a-1)}$. (ii) ±1; $\frac{1}{2}(1\pm\sqrt{5})$.

37. $\sqrt{\dfrac{5}{2}}-\sqrt{\dfrac{7}{2}}-1$; $\sqrt{\dfrac{5}{2}}+\sqrt{\dfrac{3}{2}}$.

38. (i) $(3x+y-1)(x-5y+2)$. (ii) $y^2(3x+y)(2x-3y)$.

39. $a(3b^2-a^2)=2c^3$.

40. A in 40 hours; B in 60 hours.

42. (i) $x=2$; $y=3$. (ii) $x=3$; $y=2$. (iii) $x=25$; $y=4$; $z=10$.

(iv) $x=1$; $y=\sqrt{2}$; $z=\sqrt{3}$. (v) $x=\pm\dfrac{a^2}{\sqrt{a^2+b^2+c^2}}$.

44. 0; 0. **48.** (i) 7. (ii) 4.

49. (i) 0; $\frac{1}{2}(1\pm\sqrt{5})$. (ii) $x=\frac{12}{7}$; $y=\frac{12}{5}$; $z=-12$. (iii) 3; $20\frac{1}{2}$.

50. $45br$; $75\,wi$; $150\,wa$. **52.** ma^{m-1}.

56. (i) $x=2$; $y=3$. (ii) 3; $-6\frac{1}{3}$. (iii) $\frac{1}{3}(10\pm\sqrt{115})$.

(iv) $x=\pm2$; $y=\pm3$. **60.** 4 °/o; 5 °/o.

62. $a^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})=1$. **63.** $x^2+y^2+z^2\pm2xyz=1$.

64. (i) 1. (ii) $2b$. (iii) $\frac{1}{4}(5\pm\sqrt{5})$.

(iv) $y^2+y-21=0\,(y=\sqrt{x^2-x+1})$.

66. $21\frac{9}{11}$ and $54\frac{6}{11}$ minutes past 1 o'clock.

67. 4 pairs: 240, 8; 120, 16; 40, 48; 80, 24.

70. 15; 35. **71.** 0. **74.** $x+1$; $(x-1)(2x+5)(x^4+4x^2-5)$.

75. 30 gals. **76.** $3\sqrt{3}-1$; a^n+1+a^{-n}. **77.** (i) $\frac{2}{3}$; $\frac{3}{10}$.

(ii) $x=0$; $\pm c\sqrt[4]{\{2\pm\sqrt{-1}\}}$; $y=0$; $\pm\frac{1}{5}c\sqrt{\{10\pm5\sqrt{-1}\}}$.

78. $(2x^2+3ax+7a^2)\div(x^2-6ax+2a^2)$.

79. $(A'C-AC')^2=(B'C-BC')(A'B-AB')$.

80. $\{40b-b^2\pm b\sqrt{(b^2+2400)}\}+2(b+10)$, where

$b=\frac{5}{18}\times3\cdot14159$.

81. $m^{\frac{2}{3}}-n^{\frac{2}{3}}=4$. **82.** (i) $\{4ab\pm(a+b)\sqrt{(6ab-a^2-b^2)}\}$

$\div(b-a)^2$. (ii) 0; $\pm a\sqrt{5}$. (iii) $x=\pm\sqrt{\left(\dfrac{a^2+b^2}{2}\right)}$;

$y=\pm(a^2-b^2)\div\sqrt{\{2(a^2+b^2)\}}$. (iv) $x+y=6$ or 3; $xy=3$ or 6.

85. $5x^2-1$. **87.** Identity. **88.** 12. **89.** (i) $0; -3; 1; -4$.
(ii) $-1\pm\sqrt{37}$; $-1\pm\sqrt{5}$. (iii) $0; 3; \frac{1}{2}(9\pm\sqrt{-23})$.
(iv) $-1; 1; \frac{1}{10}(-9\pm\sqrt{-19})$.
(v) $x^2y^2z^2(a^3+b^3+c^3+2xyz)=a^3b^3c^3$; a cubic Eqn. in xyz.
90. 25 minutes after 4.
92. (i) $(2a+36a^{\frac{1}{3}}b^{\frac{2}{3}})\div(a-27b)$. (ii) $(4-b)^2\div(a^2+b^2)$.
93. x^2-4. **94.** (i) $\frac{69}{20}a$. (ii) $2; 5\frac{1}{3}$. (iii) $x=5, y=6; z=7$.
(iv) $x=1; y=9$. (v) $x=1; y=2; z=3$.
96. $y(y^2+2)(y^4+4y^2+2)\sqrt{(y^2+4)}$.
98. (i) $6; -7; -\frac{1}{2}(1\pm\frac{1}{2}\sqrt{221})$.
(ii) $\{a(m+n)-2ap\}\div\{p(m+n)-2mn\}$
(iii) $x=b\pm\sqrt{(a^2-b^2+c^2)}$, &c.
(iv) $x=y=z=a^2+b^2+c^2-ab-ac-bc$.
99. $(7x+32)(x-5)$; $(nx-m)(mx+n)$.
100. 3 miles an hr.; 15 miles. **101** $ad-bc$. **102.** 0.
105. $(a^2b^2+b^2c^2+c^2a^2)(a^2bc+b^2ac+c^2ab)$
$=a^2b^2c^2(ab+bc+ca.)$
106. 10. **107.** $\frac{1}{2}P^2$.
109. (i) $\pm\sqrt{\left\{a^2-\left(-h^2\pm\sqrt{\frac{2a+h^4}{2}}\right)^4\right\}}$.
(ii) $x=1; y=625$ (iii) $\pm a; \pm\frac{1}{a}$.
(iv) $x=16; 4; y=9, -3$. (v) $x=10; y=5; z=3$.
(vi) $x=\pm4; y=\pm2; z=\pm2$, &c.
110. $9\frac{5}{8}$ miles. **116.** (i) $a+b$. (ii) $\frac{11}{2}$. (iii) $-6b$.
117. $t=x$. **118.** $5\div(x^2+1)$.
120. Each child £1920 12s, each brother £960 6s.
121. $4(ax+by+cz)$. **122.** $\sqrt{x-y}+\sqrt{y-z}$.
123. (i) $\frac{1}{2}$. (ii) $x=a; y=b$.
(iii) $x=\frac{1}{3}(2a-b-c)$; $y=\frac{1}{3}(2b-a-c)$; $z=\frac{1}{3}(2c-a-b)$.
124. $(2x+y-3)(x-11y+1)$. **125.** $x-1$.
127. $(ac+bd+bc-ad)^2$. **128.** x^2.
129. (i) $5; 7; 9$. (ii) $y=c(a^2+b^2)\div2ab$. (iii) $x=y=z=1$.
130. 480; 90. **132.** x^2+xy+y^2. **135.** $a=20; b=85$.

137. $abc=(b+c-a)(c+a-b)(a+b-c)$.
138. $A(a-b)(a-c)\div(A+B+C)$.
139. (i) $2; -\frac{7}{2}$. (ii) $1; ab$. (iii) $0; \frac{a}{3}(1\pm 2\sqrt{-2})^2$.
(iv) $x=6$; $y=5$. (v) $x=b+c$; $y=c+a$; $z=a+b$.
140. 288.
141. $x^2-\frac{x}{2}+\frac{2}{x}$; $(5+\sqrt{7})\div\sqrt{2}$.
144. $x-5$; $(2a+5b)\times$ 2nd Expression.
145. (i) -1; -7; $\frac{1}{2}(-7\pm 3\sqrt{5})$. (ii) 2; $\frac{8}{13}$. (iii) 1.
146. $\dfrac{2}{x+1}+\dfrac{3}{x+5}-\dfrac{5}{x+7}$.
147. -1.
150. 5 miles; 4 and 2 miles an hour.

ANSWERS TO UNIVERSITY PAPERS.

1.—CALCUTTA PAPERS

1858.

1. $42x^{\frac{5}{6}} - 18x^{\frac{1}{3}}y^{\frac{1}{3}} - 9x^{\frac{2}{3}}y^{\frac{2}{3}} - 14x^{\frac{1}{2}}y^{\frac{2}{3}} + 6y - 4x^{\frac{1}{6}}y^{\frac{3}{4}} + 49x^{\frac{7}{6}}y^{\frac{1}{3}}$
$+ 14xy.$

2. $\frac{7}{8}$. 3. A $31\frac{17}{23}$ days ; B $42\frac{16}{17}$ days and C $104\frac{2}{7}$ days.

March 1859.

1. $1 + x^2 - x^4 - x^6$; $x - a^{\frac{1}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{1}{3}} - a$; a^2d^2.
2. (a) $x=2$. (b) $x=15$. 3. 14 and 11. 5. 24 and 15.
6. Hyp. $=8\frac{2}{3}$; per. $=3\frac{1}{3}$. 7. Rs. 925 ; Rs. 500.
8. $4\frac{4}{5}$; $3\frac{3}{7}$ and 24.

December 1859.

2. $(x^2 - xy + y^2)^2$.
3. $(x^4 - a^2x^2 + a^4)(x^2 + a^2)(x^2 - ax + a^2)(x + a)(x^2 + ax + a^2)$
$\times (x - a)$. $\frac{b}{a}$.

4. $(x + 5)$. 5. (1) $x = \frac{8}{25}a$. (2) $x = \dfrac{ab}{a + b - c}$.

1860.

1. 0. 2. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$; $\dfrac{x + c}{(a - x)(x - b)}$; $\dfrac{a^2 + b^2}{a}$.
3. (1) $x=7$. (2) $x = -\frac{5}{6}$. 4. £2 15*s*.

1861.

1. $7x^2 - 7xy + 5y^2$. 2. $\dfrac{1}{a^4 - x^4}$.

3. $x^{\frac{3}{2}}-2xy^{\frac{1}{2}}+2x^{\frac{1}{2}}y-y^{\frac{3}{2}}$.

4. (1) $x=8$. (2) $x=3$. (3) $x=16$. (4) $x=1$; $y=3$.

1862.

1. $\dfrac{x^2+y^2}{x^2-y^2}$. 2. $a^{\frac{2}{5}}-2a^{\frac{1}{5}}b^{\frac{1}{7}}+2a^{\frac{1}{5}}c^{\frac{1}{3}}-b^{\frac{2}{7}}-2b^{\frac{1}{7}}c^{\frac{1}{3}}+c^{\frac{2}{3}}$;

$a^{\;2}-2a^{-3}b+3a^{-4}b^2$.

4. (1) $x=5$. (2) $x=16$. (3) $x=\dfrac{3ac-bc}{2b}$. 5. $\frac{2}{3}$.

1863.

2. $a^7-a^6x+a^5x^2-a^4x^3+a^3x^4-a^2x^5+ax^6-x^7$;

$x^{\frac{7}{12}}y+x^{\frac{1}{3}}y^{\frac{2}{3}}-x^{\frac{1}{4}}y^{\frac{4}{3}}-y$.

3. (1) $x=5$. (2) $x=\frac{3}{4}$. (3) $x=2$; $y=3$. 4. 24 ft.

1864.

1. $2(x^2+y^2+z^2-xy-yz-xz)$.

2. $x^{\frac{3}{4}}-3x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{2}}+y^{\frac{3}{2}}+z^{\frac{3}{2}}x^6+ax^5-a^3x^3+a^5x+a^6$.

3. $\dfrac{1}{(x-1)(x-2)(x-3)}$.

4. (a) $x=10$. (b) $x=5$; $y=11$.

1865.

1. $(x+y)^2-1$. $(u+x+y+z)(u+z-x-y)(x-y+u-z)$
$\times(x-y-u+z)$.

2. x^2+2x+3; $\dfrac{x-5}{x+5}$. 3. (1) $x+y+z$. (2) $\dfrac{1}{xyz}$. (3) 0. 4. 2.

5. (1) $x=\dfrac{a^2c+ab^2+bc^2-a-b-c}{ac+ab+bc-1}$ (2) $x=\frac{8}{25}a$. (3) $x=4\frac{1}{2}$.

(4) $x=\dfrac{4a^2c}{4b^2+c^2}$.

1866 (A).

1. -3. 2. $3a^2-ab-b^2$; $x^4+x^2+1+x^{-2}+x^{-4}$; $a^2x^4-a^{-2}$.

3. $x+3$; $xy(y^2-x^2)$. 4. (1) $\dfrac{1}{x+y}$. (2) $\dfrac{a+4}{a+5}$.

5. x^2+2x+3.

6. (1) $x=24$. (2) $x=\frac{1}{2}(a+b)$. (3) $x=\dfrac{2pr}{p^2-2pq-q^2}$. 7. 20.

1866 (B).

1. $2x^2y^2+2x^2z^2+2y^2z^2-x^4-y^4-z^4$;

$x-y$; $xy+xz+yz+z^2$.

2. x^2+2x+3. 3. $x^2+\dfrac{1}{x^2}-2$. 4. (i) $x=1$. (ii) $x=7$.

1867.

1. $\dfrac{x-1}{x+1}$; $2x+5$. 2. $\dfrac{2xy}{x^2+y^2}$.

4. (1) $x=12$. (2) $x=\frac{1}{3}(a^2+b^2+c^2)$.

5. $x=\dfrac{bc_1-b_1c}{ab_1-a_1b}$; $y=\dfrac{ac_1-a_1c}{a_1b-ab_1}$; $x=4$; $y=5$; $z=6$.

6. x^3+4c-1; $a^2-b^2+c^2-d^2$.

1868.

1. $35{\cdot}69\ldots$; $a-(b-c+d)$. 2. $-\dfrac{a^4+a^2b^2+b^4}{ab(a-b)^2}$.

3. $x=6$; $x=4$. 4. $x-1$; $(x^2+x-2)(x^2-x+1)(x^2-4)$.

5. $x=12$; $y=12$; $z=12$. 6. 23.

1869.

1. $x^2-xy+y^2+x+y+1$; $x^2-\frac{2}{3}x-\frac{2}{3}$.

2. x^2+1; $6x^3+11x^2-3x-2$.

3. (a) $\dfrac{4a^3x}{x^4-a^4}$. (b) $\dfrac{3}{a+c}$. (c) $\dfrac{3x^2+x}{4x^2+2x-1}$.

4. (a) $x=18$. (b) $x=\dfrac{a^2-b^2}{ma-bn}$; $y=\dfrac{a^2-b^2}{an-mb}$. 5. 8 days.

1870.

1. $9a^2+6ac-3ab+4bc-6b^2$;

$x^4+x^3y+x^2y^2+xy^3+y^4$; $\dfrac{a^4-a^2b^2+b^4}{a^2b^2}$.

2. x^3 ; $\frac{(2+x^2)(1+x^4)}{x}$. 3. $x=-\frac{1}{3}$; $x=\frac{6}{7}a$; $x=\frac{1}{15}$; $y=18$.

4. x^4-y^4. 5. $2x^2+2ax+4b^2$; $\frac{2x^2+3x-5}{7x-5}$.

6. 12 o'clock ; 125 miles from A.

1871.

1. $2x^5-x^4y-\frac{35}{6}x^2y^3-\frac{1}{3}x^3y^2+y^5$; $x^2-1+\frac{1}{x^2}$.

2. $\frac{2x+3}{5x-2}$; $2x(x+2)(x-2)^2(x^2+2)$.

3. (i) $\frac{2a}{a+b}$; (ii) $\frac{x-x^2}{2+x}$. 4. (i) $a=3$. (ii) $x=2\frac{1}{8}$; $y=3\frac{5}{9}$.

5. A Rs. 630 ; B Rs. 810.

1872.

1. x^2+2x-3 ; $3x-11$. 2. (i) $\frac{c^3}{x^3+a^3}$. (ii) $\frac{1}{xyz}$.

3. (i) $x=7$. (ii) $x=8$; $y=5$.

4. A Rs. 500 ; B Rs. 400 ; C Rs. 200.

1873.

1. (i) $\sqrt{(5xy)}$. (ii) $x+y$. (iii) $\frac{2x^3}{x^4-1}$.

2. $x-2a$; $3ax(x^2-a^2)$.

3. (α) $x=-\frac{4}{5}$. (β) $x=\frac{1}{2}$. (γ) $x=2\frac{1}{2}$; $y=1\frac{1}{2}$. 5.

1874.

1. (i) x^{-m}. (ii) $\frac{a^2+b^2}{a^2-b^2}$. (iii) $a-b$.

2. $1+4x-16x^3-32x^4+64x^6$; $3x^2-\frac{1}{3}xy+3y^2$.

3. $x=7\frac{1}{2}$; $x=-7$. 5. 84.

1875.

1. $b-a-c$; $\frac{x^3}{y^3}-\frac{y^3}{x^3}$; $2x^2-x-2$. 2. $\frac{1}{1-x^2}$.

3. (i) $x=\frac{6}{7}$. (ii) $x=\frac{2}{5}$. (iii) $x=\frac{3}{14}$; $y=\frac{5}{7}$. 5. 2080.

1876.

1. (1) $a+3b+2d$. (2) $8x^3$. (3) $\dfrac{4a^4}{(a^2+b^2)^2}$.

2. $2x-1$; $x^2+1+\dfrac{1}{x^2}$; $2-c+\frac{1}{2}b$.

3. (1) $x=56$. (2) $x=7$; $y=9$. 5. 4 days.

1877.

1. $\dfrac{4x^4+8}{x^8+x^4+1}$; $2b^2c^2+2a^2b^2+2a^2c^2-a^4-b^4-c^4$

x^2-x-19.

2. $x=4\frac{1}{3}$; $x=\frac{5}{3}$; $x=1$; $y=2$; $z=3$. 3. x^2-4x+3.

4. $a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}-$ &c. 10 0498756 6. 40.

7. $\frac{23}{25}$ cwt. ; 2 cwts. ; 3 cwts.

1878

1. $x+y+z+xyz$ 2. $ax+by+cz$. 3 0.
4. $x=-3$; $x=6$. 5. 1.

1879.

1. $a^{3n}+x^{3n}$; $x+\frac{1}{2}$. 2. $x^{2^{n-1}}-y^{2^{n-1}}$; 2.

3. (a) $x=\dfrac{m(m+2k)}{2(m+k)}$. (b) $x=y=3$. (c) $x=4$; $y=10$.

4. In favour of the latter. 5. 11 days.

1880.

1. 1. 2. $x-a$; $(3x-7a)(x-3a)(x-a)^2$.
3. (a) $x=9$. (b) $x=11$. (c) $x=3$ and $y=2$.

5. $a^{-\frac{3}{5}}-x^{\frac{7}{5}}-x^{\frac{4}{5}}-a^{\frac{4}{5}}$. 6. 3 miles ; 8 miles.

1881.

2. $\dfrac{1}{x(x-a)(x-b)}$ $(x+6a)(x-11a)$; $2(1+c)(1+a)(a-c)$.

5. (1) $x=5$. (2) $x=\frac{3}{10}$. (3) $x=\dfrac{c(c+b)}{a(a-b)}$; $y=\dfrac{c(a-c)}{b(a-b)}$.

6. A in 120 secs. and B in 130 secs.

1882.

1. $x^{n-1}+ax^{n-2}+a^2x^{n-3}+\&c.+a^{n-2}x+a^{n-1}$; x^6-a^6.
2. $(x+6)(x+7)$; $(x-6)(x+7)$; $(7x+8y)$
$\times(49x^2-56xy+64y^2)$.
3. G.C.M. $=x-8$; and L.C.M. $=(x-8)^2(x+9)(9x^2-100)$.
5. (i) $x=3$. (ii) $x=5$. (iii) $x=8$; $y=-15$.
6. $2\frac{5}{8}$ miles from P the starting point.

1883.

1. $a-b+c$. 2. $2x-y$. 3. $\dfrac{a^3}{(a-x)(a^3+x^3)}$.

$\dfrac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$.

5. (a) $x=\frac{1}{2}\{\sqrt{(a^2+2b^2)}+\sqrt{(a^2-2b^2)}\}$; and
$y=\frac{1}{2}\{\sqrt{(a^2+2b^2)}-\sqrt{(a^2-2b^2)}\}$.
(b) $x=10\frac{3}{8}$. (c) $x=10$. (d) $x=-28$; $y=10$ and $z=9$.
6. 54 and 646.

1885.

1. $b-pa+q$; $\frac{8\,3\,1}{3\,3\,8\,2}$; $\frac{2\,8\,2\,3}{3\,3\,5\,2}$.
2. $(x-a)(x^2+ax+a^2)$; $(x-a)\left(x-\dfrac{1}{a}\right)$;
$(x^2+a^2+ax)(x^2+a^2-ax)$; $x^{10}-x^5-x^4+1$.
3. (i) $x^{2n}+2$. (ii) $n(n-1)$.
4. (i) $x=\dfrac{1}{1-a}$. (ii) $x=\dfrac{81}{a}$. (iii) $x=\pm1$.
(iv) $x=\dfrac{mp-nq}{aq}$; $y=\dfrac{mp-nq}{aq}$.
5. $1-\frac{1}{2}x-\frac{5}{8}x^2-\frac{5}{16}x^3$. 6. £52 ; 52*s*.
7. 4 oranges for 3*d*. ; 512.

1886.

1. $\dfrac{(m-s)(m-n)}{2m-s-n}$; $\dfrac{x+1}{(x-1)(2x-1)}$.
2. $(9x^2-1)(9x^4-1)(x^2-3)^2$. 3. $\dfrac{a^2+2ab-b^2}{a^2-b^2}$.

4. (a) $\dfrac{b^2+ac}{b^2+c^2}$. (b) $x=5$. (c) $x=10$; $y=20$; $z=5$. (d) $x=\frac{1}{4}a$.

5. Debts = £8,000. Assuming debts = Assets.

6. $27\frac{3}{11}$ min. after 5.

1887.

1. $x^2+y^2+z^2-xy+xz+yz$; $(x^2+12x+31a^2)^2-(4a^2)^2$.

2. $(a^2+ax+x^2)(a^2-ax+x^2)(a^4-a^2x^2+x^4)$;
$(x^2+2a^2-2ax)(x^2+2a^2+2ax)(x^2+2a^2)(x^2-2a^2)$.

3. $a+b+c$.

4. (1) $x=3\frac{1}{3}$. (2) $x=2$; $y=3$. (3) $x=1$; $y=3$; $z=5$.

5. 600.

1888.

2. $2x(a+b)$. 3. (1) $ab-ac+bc$. (2) $x^2+(y+z)x+y^2+z^2$.

4. (1) $x=\frac{14}{13}$. (2) $x=-2\frac{30}{73}$. (3) $x=4$; $y=5$.

5. Length = 10 yds.; breadth = 7 yds. 6. $x^2+ax-2a^2$.

1889.

1. $x=5$. 2. $\dfrac{x^2}{2}-2x+\dfrac{a}{3}$. 3. 0.

4. $\dfrac{3(x-3a)(x-4a)}{2(x+3a)(x+4a)}$. 6. 1.

1890.

1. x^2-x+1. 2. $3x-4+\dfrac{1}{2x}$.

3. (i) 5. (ii) $\dfrac{a+b}{2}$. (iii) $\dfrac{1}{5}$; $\dfrac{1}{7}$. 4. 1220.

1891.

2. (a) 2. (b) $(ab-ad)\div(a+b-c-d)$. 3. 22; 24.

4. $4a^2-3ab+b^2$.

1892.

1. 94. 2. $(2a-3b+c)\div(2a-3c)$. 3. (a) 4. (b) $a-2b$.

4. 12′ past 4. 5. 1.

1893.

1. x^2-x+1. 2. $2x^3-3x^2+x-4$.

3. (1) 2. (2) $-a$. (3) 3; 4. 4. $485\frac{1}{7}$; $348\frac{2}{7}$. 5. 5.

II.—BOMBAY PAPERS.

1859.

1. $4\frac{13}{15}$. 4. a^3-64b^3. 5. $x^{-\frac{2}{3}}+x^{-\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$.
6. Art. 70. 9. (i) $5\frac{5}{11}$. (ii) 2. (iii) 1 ; $1\frac{8}{21}$.
10. 22 yrs. hence. 12. 240 ; 120.

1860.

1. $\frac{8}{5}$. 4. $\frac{x^2}{2}+\frac{2x}{3}+\frac{2}{x^2}$. 6. $2a+3x$; $4x(2a+3x)$.
7. 1 ; x ; $(x^{3n}-x^{2n}+2x^n)\div(x^n-1)$. 8. $7\frac{52}{105}$; $1\frac{3}{4}$. 9. $2m$.

1861.

1. a. 3. $a-2p$. 4. m.
5. (1) $1\frac{1}{2}$. (2) Quadratic. (3) $-a$; $-b$.
6. 9 h. ; 10 h. 3′ ; 11 h. 6′ ; 12 h. 9′.

1862.

4. 84 yrs. 5. $(x\pm y)^2$. 6. 32 ; 24. 7. $x+3$(G.C.M.).
8. *See* Art. 64. 9. $16\frac{4}{11}'$ and $49\frac{1}{11}'$ past 3.
10. $y=-\frac{77}{301}$; &c.

1863.

1. $7\frac{1}{2}$. 3. $a-b$. 4. $(a+1)(a+4)(a^3-1)$.
5. (*b*) $4x\sqrt{x^2-1}$. 7. $a^{\frac{3}{2}}-b^{\frac{2}{3}}+c^{\frac{3}{2}}$.
8. (*a*) 11. (*b*) $\frac{1}{6}$; $\frac{1}{7}$. (*c*) $\frac{5}{4}$. 9. $1\frac{7}{8}$ hrs. 10. 12.

1864.

2. $-\frac{7}{12}$. 4. a^5-b^5. 5. a^2-x^2.
6. (i) $(a-b)(a-c)\div abc$.
8. (1) $\frac{26}{5}$. (2) $x=(m^2-n^2)\div(ma-nb)$. 9. $\frac{27}{4}$.
10. 8 days.

1865.

1. $3a-6c$. 2. (i) 9. (ii) 9.
5. $\frac{1}{abc}$; $(x^4+x^2a^2+a^4)\div a^4$. 6. $1+\frac{x}{2}-\frac{x^2}{8}$.
7. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b$. 8. (i) 1. (ii) 16. (iii) $(a^2-b^2)\div 6a$.
9. 30 hrs. 10. 72 *m*.

1866.

1. 536. **3.** $3a^2+4ab+b^2$. **5.** $x+2$. **6.** $\dfrac{4}{3(x+1)}$.

7. (i) $2x^2-x+1$. (ii) $x^3-\dfrac{x}{2}-\dfrac{2}{x}$.

9. (i) 1. (ii) $12\frac{8}{203}$; $6\frac{36}{203}$. **10.** 1*s*; $1\frac{2}{3}$*s*. **11.** 100*m*.

1867.

1. (i) 6. (ii) 60. **2.** x^2-y^2. **3.** $x^3-xy^2+y^3$.
5. (ii) $5\div2(1+2x)$. **6.** $x^{\frac{1}{6}}$. **8.** (i) 4. (ii) 3. (iii) 6; 10.
9. $49\frac{1}{11}'$ past 3; $32\frac{8}{11}'$ past 3. **10.** Rs. 1,500; Rs. 500.
11. $3\frac{6}{11}m$ from B; $14\frac{7}{13}m$ from T.

1868.

2. $x-y$. **5.** (i) 1. **6.** (i) $\frac{4}{3}$. (ii) $ac\div b$. **7.** £$17\frac{1}{2}$.
8. (i) $x=(cn+bd)\div(an+bm)$. (ii) $2a$; $-2b$.
9. $2\frac{1}{2}$ *m*; 2 *m*; 5 *m*.

10. (i) $2x^2-3ax+4a^2$. (2) $2a^{\frac{1}{2}}-3b^{\frac{1}{3}}+4c^{\frac{1}{4}}$. **11.** $1+\sqrt{3}$.

1869.

2. $7x^2-y^2-2xy$. **4.** $(a+3d+2b+c)(a+3d-2b-c)$.
5. $2a^2x(2a-3x)$. **6.** $(2b-a)\div(a+b)$. **7.** $x^3-2x^2y-y^3$.
8. (i) 9. (ii) $3\frac{3}{8}$. **9.** 56; 203. **10.** $x=1\frac{219}{247}$.
11. 40; 24 days; £$6\frac{1}{4}$; £$13\frac{3}{4}$.

1870.

1. $3\frac{7}{8}$. **4.** G.C.M. $(x-y)^3$. **5.** $(x+y)\div x$.
7. $x^{\frac{1}{2}}+3x^{\frac{1}{3}}y^{\frac{1}{3}}-2y^{\frac{1}{2}}$ **8.** (*a*) $17\frac{403}{468}$. (*b*) $x=4$.
(*c*) $x=1$; $y=1\frac{1}{2}$.

9. 1,000 men. **10.** $13\frac{1}{2}$ *m*; 9 *m*.

1871.

2. $(x^2+a^2)\div x$. **4.** x. **5.** $a^2-a+\frac{1}{4}$. **6.** It is.

7. 1; $-\dfrac{50}{9}$. **8.** -1; 2; -2; 1. **9.** 240; 120. **10.** 800 men.

1872.

3. $(y^2-4)^2$. **6.** x^2+2x+3. **7.** $a+\frac{1}{a}+1$; $\frac{x}{y}-\frac{2y}{x}$.

8. (1) $2\frac{2}{5}$. (2) $4\frac{1}{2}$. (3) 12. (4) $10\frac{1}{2}$. **9.** K gets Rs. 3,500.

10. Whole is Rs. 3,840. **11.** £15. **12.** 400 inches.

1873.

2. 18. **5.** $x+1$; $120xy(x^2-y^2)$. **6.** $2x^2+3xy+y^2$.

7. (i) 41. (ii) 11; 9. **8.** 70; 30. **9.** 91:

1874.

2. (i) 1. **3.** (a) $3a^2x(x+a)$.

4. (a) x^3-2x^2+3x-4. (b) $2x+5y$. **5.** 26; 27.

6. 17 *m*; 4 *hrs*. **7.** $(am+bn+cp)\div(a+b+c)$.

8. $x=\frac{30}{11}$; &c. **9.** 324.

1875.

2. $a-b^2$. **3.** G.C.M. is $a-b$. **4.** $1\div(2a^2-1)$.

5. $a-b+1$. **6.** (i) $\frac{1}{2}$. (ii) $\frac{4}{9}$. (iii) 4; 5; 6. **7.** Rs. 600.

1876.

1. (i) -54. (ii) 0. **2.** (i) 1. (ii) 1. **3.** $2y^2$.

4. (i) $(x-3y)(x+2y)$. (ii) $(x-y)(x+2y)(x-2y)$; $1+x^{\frac{1}{2}}$.

7. $2x^2-3x^{-1}+4x^{-4}$; $2x^3-x^2-3x-3$.

8. $x=1$. **10.** 1*s*. 4*d*.

1877.

1. $x^2-3xy+2y^2$. **6.** (i) 0; $\frac{a+b}{2}$. (ii) 0; $-8\frac{1}{2}$.

(iii) $113y=147x$.

7. Rs. 25. **8.** 35 or 53.

1879-80.

1. $a^2+b^2+c^2$. **2.** $\frac{x}{y}+\frac{y}{x}-\frac{1}{2}$. **3.** $x-2y$. **4.** 1.

5. $y=(c^2-bc+a^2)\div(b+c)$. **6.** Rs. 8,000.

1880-81.

3. $x-2$. **4.** 1 ; 2 ; 3. **5.** $54\frac{6}{11}'$ past 10 or 11. **6.** Rs. 80 , Rs. 60.

1881-2.

3. $\frac{x}{y}-\frac{y}{2x}-\frac{1}{2}$. **4.** a^2-x^2 ; $(a-x)(a^4-x^4)$. **5.** 23.

6. (1) 9. (2) 20. (3) 9 ; 20.

1882-3.

3. x^2-3x+4. **4.** $x-4+\frac{2}{x}$. **5.** 0 ; $\frac{a+c}{2}$. **6.** 30 : $7\frac{1}{2}$; 6.

7. Rs. $83\frac{1}{3}$.

1883-4.

2. Apply formula of Art. 50 ; $(a+b)(b+c)(c+a)$.

3. $10\frac{5}{13}$ hrs. **4.** 12 ; 35 ; 5 ; 75. **5.** $4\frac{1}{13}$; 2.

1884-5.

1. $16(x^4-x^2y^2+y^4)-8a^2(x^2+y^2)+a^4$.

3. $(a-b)^2+(b-c)^2+(c-a)^2$; $(5x+1)(2x-5)$

4. $(x-a)^2$. **5.** 91 **6.** $4\frac{7}{8}$ m

1885-6.

1. x^8-16y^8. **2.** Second expression $\times (a+5b)(a+6b)$.

3. $\frac{x-3}{x-4}$; $\frac{1}{a+2}$. **4.** (i) 6.

(ii) $x=pr\div(p^2-q^2)$ · $y=qr\div(p^2-q^2)$. **5** £$33\frac{3}{4}$.

1886-7.

1. (i) $2(x+4)$. (ii) 1. **2.** p^3-3p. **3** G.C.M. is x^2+x-3.

4. $\frac{x^3}{4}-\frac{x^{\frac{1}{2}}}{3}-\frac{1}{2}$; $2x^2-3x+1$. **5.** (i) $2a$. (ii) 3 ; 2. **6.** $1\frac{9}{16}m$.

1887-8.

2. (i) 0. (ii) -1. **3.** $(x-4)(x-3)(x+5)$.

4. a^2+b^2. **5.** 8. **6.** 640.

1888-9.

1. (ii) $(x^2-y^2+pxy)(x^2-y^2-pxy)$.
3. $1+a+a^2+\ldots\ldots\ldots\ldots+a^9$. **4.** -1. **5.** $a+b$; $a-b$.
6. 253. **7.** 76; 30.

1889-90.

2. a. **3.** $\left(\frac{p}{q}\right)^{m+n}$. **5.** (i) 1. (ii) 5.
6. $(10x+y)+(10y+x)=4(x+y)+7(x+y)$. **7.** 84; 63.

1890-91.

1. 0. **2.** (b) 1. **3.** $\frac{x}{a}-1+\frac{a}{x}$. **4.** $(x-1)^3(x+1)$.
5. (i) $b(c+a-b)\div a$. (ii) 1; $\frac{1}{2}$; $\frac{1}{3}$. **6.** 9; 6 gals.

1891-92.

1. (i) $(x-3y)\div(x+3y)$.
(ii) Apply $a^3+b^3=(a+b)(a^2-ab+b^2)$.
2. (b) $p^{p+q}\div q^{p+q}$. **3.** $x+4$ is the G.C.M. **4.** $x=1$.
5. 7. **6.** £4,680; £4,720. **7.** 3.

1892-3.

1. 189. **2.** x^2+x+1.
3. (i) $(a+b)\div ab$. (ii) $(x+1)^2\div(x+2)$.
4. (i) $1\frac{1}{12}$. (ii) 3; 2. (iii) $4\frac{1}{2}$. **5.** $\frac{3}{8}$.
6. 220 yds.; 176 yds. per minute.

III.—MADRAS PAPERS.

1857.

1. 0. **2.** $x-\sqrt{xy}+y$. **3.** (1) 15. (2) ± 2; ± 5; ± 6.
4. 200 m; $33\frac{1}{3}$ m.

1858.

1. 59. **4.** $x=\left(\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right)\div\left(\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}\right)$.
5. 48.

1859.

2. $-\frac{3}{x}+\frac{15}{4}-\frac{45x}{16}+$ &c.

4. (i) $(a^2-b^2)\div(4a-b)$. (ii) 37 ; $21\frac{1}{3}$; $11\frac{1}{3}$.

5. $7\frac{1}{5}$ days.

1860.

1. $\pm(a+b)$. **2.** x^2-y+5y^3. **5.** $x=b$.

6. $x=\frac{1}{2}(3+2\sqrt{2})$; $y=\frac{1}{2}(3-2\sqrt{2})$. **7.** 35 yrs.

1861.

2. $1\frac{21}{40}$. **4** $(a+x)^3(a-x)$. **5.** $\frac{a}{2}-\frac{b}{3}$.

6. $\frac{4ab}{a^2-b^2}$; ∞ ; $-\infty$, 1, 0. **7.** 52. **8.** 5 sov. ; 2*s*.

9. (1) 1. (2) $4abc\div(ab+ac+bc)$. (3) abc. (4) $\frac{3}{4}$.

1862.

2. $a^2x^2+3bx+c$. **3.** 1 ; 0 ; 2. **5.** $4mx+6ny+3pz$.

6. (1) 7. (2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$. (3) $\frac{3}{2}$; $\frac{4}{3}$. **7.** 132 ; 108 ; 6 hrs.

8. 46 ; 80.

1863.

3. $a^2+b^2+c^2$. **5.** (1) x^3+7-2x. (2) $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

7. (1) 1. (2) $(a^2c+b^2a+c^2b-a-b-c)\div(ab+ac+bc-1)$.
(3) $x=(b_1c-bc_1)\div(ab_1-a_1b)$. (4). $\cdot 4$; $\cdot 1$.
(5) $\frac{1}{2}$; 1 ; $\frac{1}{3}$

8. Rs. 3,640. **9.** $\frac{5}{9}$.

1864.

1. (1) 2. (2) 2. **2.** 1. **4.** $(a-b)x-(a+b)y$.

5. $a+2\sqrt{b}-3\sqrt[3]{c}$; 3·07 **6.** $a-2b$.

7. (1) $(x-3)\div(x+3)$. (2) $4a^3\div(a^4-x^4)$. (3) 1.

8. (1) $\frac{\sqrt{a}}{3}$. (2) a ; b. (3) $4\frac{4}{5}$; $1\frac{5}{7}$; $1\frac{1}{5}$. **9.** 13.

February 1865.

3. (1) $a+\frac{1}{a}-1$. (2) $a^{-2}+a^{-1}b^{-1}-b^{-2}$. **4.** $x+2$.

5. $12x^4-20x^3+5x^2+5x-2$.

7. (1) $(3a^2+2b^2)\div 5a(2a+3b)$. (2) $2\sqrt{a^2-x^2}\div x^2$.

8. (1) 10. (2) 1. (3) 9. (4) 2 ; 5 ; 10.

9. $am(2m+n)\div 2(m^2-n^2)$.

December 1865.

1. x^4+9x^2+81. 2. (1) $x^2\div(x-a)$. (2) $2c\div(a-c)$.

3. (1) $(b-a)(a+b+c)(a+b-c)$.
(2) $(a+b)(c+a-b)(c-a+b)$.

4. $\cdot 1$. 6. $(x+1)(x-4)(x+3)(x^2+4x+9)$.

7. (1) $-\frac{3}{4}$. (2) 72. (3) $x=(b+c)\div 2a$; &c.

8. 32 ; 40 ; $53\frac{1}{3}$.

1866.

1. $2(x^4+1)\div(x^6-1)$. 2. $(a+b)x^3+(b+c)x^2+(c+a)$.

3. $a^2+a-\dfrac{1}{a}+\dfrac{4}{a^2}$. 4. x^2-x+1. 7. (1) 1. (2) 8 ; 4 ; 5.

1867.

1. $a^3+(m+n)a^2+2mna+1$ 2. $2\{(a+m)(c+n)+bd\}$.

3. (a) $3abc\div(a+b)$. (b) $a+b$. 4. x^2-x+1.

6. $(2x+5)(x^2-4x+3)(x^3+4x^2-5)$.

7. (a) 4. (b) $abc\div(a+b+c)$.
(c) $1\div(a+b+c)$; $2\div(a+b+c)$; $3\div(a+b+c)$.

8. (i) $\{ab-c(n+1)(a-b)\}\div\{b-(n+1)(a-b)\}$.
(ii) $\{cn(a-b)+a(c-b)\}\div\{(n+1)(a-b)\}$.

1868.

1. $c-b$. 2. (2) $(a-b)(b-c)(c-a)(a+b+c)$.

4. (1) $1\div(x^2-x+1)$. (2) 4. 5. $x-1$. 6. 1.

7. (1) $\frac{1}{9}$. (2) $a\div(a+b)$; $b\div(a+b)$. (3) 16 ; 36.

8. 18 ; 12 ; 15.

1869.

1. $1-a^2$. 2. $x^8-ax^7-a^3x^5-a^4x^4-a^5x^3+a^7x+a^8$.

4. (1) 1. (2) $2\div(x+2)(x+3)$. 5. $(a-d)(b-c)$.

9. (1) 14. (2) $1\div(1-a)$. (3) $x=bc(b-c)$; &c.

10. $10m$.

1870.

1. $x^2+xy+y^2+(a+b+c)(x+y)+ab+ac+bc$. 2. 1.
3. (1) $(c+x)\div(1+ax)(1+bx)$. (2) $(2^n-1)\div 2^n$.
4. $(x+1)^2$. 5. $(a+b-2c)\div 2(b-c)$.
7. $y-\dfrac{y}{mn}-mn+1$.
9. (1) $4\frac{1}{6}$. (2) $4an\div(n^2+4)$. (3) $x=(b+c)\div 2a$; &c.
10. $3m$; $14\ m$.

1871.

1. (i) $a+b+c$. (ii) $\dfrac{b}{a}$.
2. (i) 1. (ii) $2x^5\div(x^6-1)$. (iii) $x\div(c+bx)(a+c-x)$.
3. $(x-c)^2$. 4. $x-\dfrac{1}{x+2}$. 6. x^4+2x^3-1.
8 (i) 1. (ii) 5 : 2. (iii) $x=y=z=1$. 9. 12*s.* ; 28*s.*

1872.

1. $3x^2-2x(a+b)+a^2+b^2$.
3. (i) $(2x^2-2x+5)\div(1-x^2)(1-2x)$.
(ii) $(cx-a)\div(ax-b)(bx-c)$.
5. $3x+1$. 6. $\left(a\pm\dfrac{1}{a}\right)^2+\left(b\mp\dfrac{1}{b}\right)^2$. 7. $3x^2-2x-1$.
8. (i) 11. (ii) 0 (iii) $x=3(3b-2a-c)\div 2a$.
9. $5m$; $3m$; $40m$.

1873.

1. $(a+b)(a+b+c)$; a^2-b^2. 4. a^2-2a+2.
5. (i) 4. (ii) $4\div(x+1)(5x+1)$. 6. $(a+b)^2(c-d^2)$.
9. (i) 7. (ii) $ab\div(an^2+bm^2)$. (iii) $x=y=z=(a+b)\div c$.
10. $1\frac{2}{3}m$.

1874.

3. 2. 4. a^2-3a+5. 5. (1) 3. (2) $2(ac+bd)(ad+bc)$.
7. $(a-b)(ab+1)$.
9. (1) 2. (2) $1\frac{3}{8}$. (3) $2(a-1)\div(a-2)$; $(a+1)\div(a-2)$; $(3a-1)\div(a-2)$. 10. $3m$.

1875.

1. 0. **2.** 0. **3.** $4(a+b)^2$. **5.** G.C.M. is $2a^2-3a+2$.
6. G.C.M. is a^2+a+1. **7.** (i) $\sqrt{a+x}$. (ii) $c\div a$.
8. $4x^2(a+b)^2$. **9.** (i) 2. (ii) $ab\div(a+b)$. (iii) 1; 2
10. $3\frac{1}{2}m$; $4\frac{1}{2}m$.

1876.

1. $a+c-b$.
2 $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.
4. (ii) $q=3$; $(x-2)(x-3)(x-4)$.
5. (i) $4(al+bm+cn)$. (ii) 0.
6. $(ax-cy)^2-(bx-dy)^2$. **7.** $a+b+c$; $(a+b+c)^2$.
8. $4xy$. **9.** (i) $(ab-cd)\div(c+d-a-b)$.
(ii) $e\div(a-b)$. (iii) $x=b^2\div 2a$; $y=(2a^2+b^2)\div 2a$.
10. $d(a-b)\div(b-c)$; $d(a-b)(a-c)\div 2(b-c)$.

1877.

1. $4(a-b)^2(a+b+1)$. **2.** $a+b+c$.
3. x^2-3x-2; $x^2-\frac{x}{2}+\frac{2}{x}$. **4.** $\pm\sqrt{\pm 5}$. **5.** 60.

1878.

1. 2. **2.** $(x-1)^4(x+1)^4(x+2)^2(x-2)^2$.
4. (1) -2. (2) $x=c-b$; &c. **5.** $30m$; $6m$.

1879.

1. 0. **2.** $\{(x+y)^2-z^2\}^2+4(x+y)^2z^2$.
4. (1) 9. (2) $2\div(1-a-b-c)$. (3) $x=1$; $y=1$; $z=0$. **5.** 8′.

1880.

1. $a^2-(2b-3c)^2$. **2.** 1. **3.** 3.
4. (1) $(x-1)^2$. (2) $x+2+\frac{3}{x}$.
5. (1) 13. (2) 34. (3) $x=a\div(a+b)$; $y=b\div(a-b)$.
6. $102\frac{8}{21}m$.

1881.

2. $\sqrt{a}+\sqrt{2b}+2\sqrt{2c}$. **3.** 1.
4. (i) $2\frac{1}{7}$. (2) $\frac{1}{2}$; $\frac{1}{3}$. (3) $x=(2c+a+b)\div 4$.
5. 30; 20; 50. £72; £36; £18.

1882.

1. 2556. 2. $x^2-2xy+3y^2$. 3. ab.

4. $\frac{x^2}{y^2}-\frac{x}{y}+1-\frac{y}{x}+\frac{y^2}{x^2}$. 5. (1) $p-q$. (2) $5\frac{3}{8}$. 6. $\frac{1}{7}m$.

1883.

1. $a+b+c$. 2. $1+\frac{2x}{a}+\frac{4x^2}{a^2}+$, &c.

3. $(2x-y)(2x+y)(3x^2+y^2)$.

4. (a) $(a^2-b^2)(p^2-q^2)^2 \div (p^2+q^2)^2$.

(b) $(x-y+z)\div(x-4y-4z)$. 5. $\frac{2x}{3y}-\frac{3y}{4z}-\frac{4x}{5z}$.

6. (a) 5. (b) $x=b\div a(b-a)$; $y=a\div b(a-b)$;

(c) $x=bc(1+a)\div(b-a)(c-a)$, &c.

7. Re. 1; Rs. 100.

1884.

1. 0. 2. G.C.M. is a^2+2a+3. 4. $3a^2-8a-16$.

5. (i) $\frac{1}{10}$. (ii) $x=a$; $y=b$. 6. $5m$; $15m$.

1885.

1. $a^2+b^2+c^2$; *See* formula. 2. G.C.M. is x^2+2x+3.

3. $x^2-ax+2a^2$. 4. (1) 4. (2) $x=(m^2-n^2)\div(am-bn)$.

5. $27m$.

1886.

1. $6(x+4)$. 2. $(3x-2y)\div(2y+x)(2x^2+y^2)$.

3. $\frac{x^2}{2}-\frac{2x}{3}-\frac{3}{4}$. 4. (1) $b(a+c)\div(a-c)$. (2) $\frac{2}{3}$; $-\frac{3}{4}$.

5. (1) 20; 60; 30. (2) 10.

1887.

2. $x-4$. 3. $\frac{1}{x+y}$.

4. (1) $-1\frac{1}{2}$. (2) $x=(b^2-a^2+c^2)\div(bm-an-cn)$. 5. 63.

1888.

1. $a+2b+3c$. (2) $(x+6)(x+2)(x^2+8x+10)$.

2. G.C.M. is $x+3$. 4. (1) $-2ab\div c$. (2) 6; 7; 8.

5. 8 yds.; $7\frac{1}{3}$ yds.

1889.

1. (1) $2a$. (2) x^{2abc}. (3) $(a^4-b^4)^m$. 2. $x^2-(p+q)x+q^2$.

3. (2) $1-\frac{xy}{2}-2x^2y^2$. 4. (1) $(x-1)^2$. (2) -3.

5. (1) $-5\frac{1}{2}$. (2) 4. (3) 1; -2; 3. 6. 96; 70.

1890.

1. (1) $-(m+n+x+y)$. (2) $1\div x^{a^2+b^2}$.
(3) $x^8-x^6y^2-x^2y^6+y^8$.

2. $(2x-1)(x+2)$; $(4x^2-3x+1)(3x-1)(2x-1)(x+2)$.

3. $x^2-xy-xz-yz$. 4. (1) 4. (2) x.

5. (1) $3\frac{1}{2}$. (2) 6; 11; 6. 6. 58; 42.

1891.

2. $2+4x+8x^2+16x^3+\&c$ 3. (a) $x^3-5x^2+13x-14$.

4. $(7x^2-59x+18)\div(x+1)(x-2)(x-6)$.

5. (a) 9. (b) 7; 3. (c) $x=12mab\div(a+b)$. 6. 420; 255.

1892.

2. $(a-b)(b-c)(a-c)(a+b+c)$. 3. -1.

4. $\frac{2x}{3a}+\frac{3a}{2x}+\frac{3ax}{b^2}$. 5. x^2+x+1.

6. (i) $x=y=5$. (ii) 9; $-1\frac{4}{51}$. 7. $5'$, $5\frac{1}{3}'$.

1893.

1. (a) $12abc$. (b) apply $a^2-b^2=(a+b)(a-b)$.
(c) $x^2+5(a-1)x-b$.

2. (a) $(x+1)(x-1)(x+1-xy+y)(x-1-xy-y)$. (b) $a-b$.

3. $2x^3-4x^2+x-1$. 4. $x^{2m+2}+3x^m-5y^{m-2}$.

5. (1) $x=abc(bc-ab-ac)\div(b^2c^2-a^2b^2-a^2c^2)$. (2) 2; $4\frac{6}{13}$.

6. 24.

1894.

1. (1) y^2. (2) $6xyz$. 2. $(a+bx)(x^2-x+1)$.

3. $x^3-x^2a+2xa^2+a^3$.

4. $(x^6-a^6)(x^3+a^3)$; H.C.F. x^2+2x+3.

5. $(a-b+c-d)(a-b-c+d)(b-c+d-a)(b-d-c+a)$
$/(a-b)(b-c)(c-d)(d-a)$

6. (1) $2\frac{1}{3}$. (2) $x=3$; $y=2$; $z=1$. (3) 11 or 2

7. 13/21. 8. $(a+4b)\div3$.

1895.

1. $(3x-2)(2x-3)$. **2.** (1) 1. **3.** $x+4$.
4. $x^3+x+2-2/x^3$
5. (1) $-1\frac{2}{5}$. (2) $y=-ab^2c/a^2+b^2)$. (3) $45 \div 47$; $-\frac{1}{3}$.
6. 1st year Rs. 1,296 ; 2nd year Rs. 2,000.

1896.

1. (1) 0. **2** $9x^2+2x+3$. **3.** x^6-a^6.
4 $2x+1-3/x-1/x^2$. **5.** (1) $-7\frac{2}{3}$.
(2) $x=1$; $y=\frac{1}{3}$, $z=\frac{1}{2}$ (3) $\frac{40}{43}$; $-\frac{2}{3}$. **6.** 5 miles.

1897.

2. (1) *Vide* Q. (2) Year 1869, Madras. (2) $a+b+c-3x$.
3. (1) $(2x^2+1)(2x^2-1)(2x^2+1+2x)(2x^2+1-2x)$.
(2) $(a+c+b-d)(a+c-b+d)(b+d+a-c)(b+d-a+c)$.
(3) $(x-y)(c-y-1)$.

4. (1) $\dfrac{a-b}{b-c}$. (2) $\dfrac{4b}{a}$.

7. (1) $x=7$. (2) $x=-2$ or $-\frac{89}{75}$.
(3) $x=-1$, $y=2$, $z=-3$. (7) 13 as. 4 ps.

1898.

1. (a) x^4-x^2-1. (b) a^2-ab+b^2. **2.** (1) $(a+b+c)$ $\times(a+b-c)(c+a-b)(c-a+b)$ (2) $(a^2-ab+b^2)(a^2+b^2+a+b)$.
3. x^2-2x+2. **4.** (a) $3 \div (x-1)(x^2+1)$.
5. $3a^2-ab+5b^2$. **6.** (1) 19. (2) $x=6+a$; $y=6-a$.
(3) 3 or $\frac{2}{3}$. **7.** $L=10$ yds. ; $B=7$ yds.

1899.

1. $(x+2a)(x+3a)(x-2b)(x-3b)$; $x^2(a+1)+x(a^2+1)+a^2$.

2. (1) $(a^2+1)(b^2+1)(a+1)(a-1)(b+1)(b-1)$.
(2) $(a-b)(b-c)(a-c)$.

3. G.C.M. is x^2-5x+1. **5.** $x^3+\dfrac{1}{x^3}+3\left(x+\dfrac{1}{x}\right)$.

6. (1) $\dfrac{2}{17}$. (2) 5 or $\dfrac{6}{5}$. (3) 2 ; -2 ; 5.

7. $4\frac{1}{2}mi$; $4mi$; $3\frac{3}{4}mi$.

March 1900.

1. (2) $2x^4-5x^3-4x^2+x+2$.

2. (1) $(1+a)(1-a)(b+c+ab-ac)(b+c-ab+ac)$.
   (2) $(a-b)(b-c)(c-a)(ab+ac+bc)$.

3. x^2+7x+1. 4. (1) $4 \div (x^4-16)$. (2) $2c$.

5. $a^2+5ab-3b^2$. 6. (1) 4. (2) $1\frac{1}{6}$ or 4. (3) $1\frac{1}{2}$; $-\frac{2}{3}$; 1.

7. Rs. 200 ; 90 men.

December 1900.

1. (1) m. 2. $(a+b)(x+b)(x-a)(x+a-b)$.

3. x^2-7x+4. 4. (1) 1. (2) $(a+b)^2(a^2+b^2)^2$.

6. (1) $3\frac{3}{8}$. (2) 2 or $4\frac{6}{13}$. (3) 8; 4; 5.

7. 52 lbs.

May 1890

2. (b) $16ab^3$. 3. $a(2-a)-a-1$.

4. $\frac{a}{3}$. (2) $\frac{a}{b}$; $\frac{b}{c}$. 5. 100 m; 10 *A.M.*

May 1891.

1. $x^2+y^2+1-xy-x-y$.

2. (a) (i) $3(x-y)(y-z)(z-x)$. (ii) $(x+y)(x-y)(3x+y)$
   $\times(3x-y)$. (b) 8.

3. (b) $\{a^2x^4-(bx+c)^2\}(a^2x^2-c^2)$.

4. (i) $x=(a+b)(b^2-a^2)\div(a^2+b^2)$. (ii) $1\div 2(a+b)$; $-1\div 2(a+b)$.

5. $5\frac{5}{11}'$ past 4.